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p and hp Finite Element Methods: Mathematics and Engineering Practice

Conference in honor of the 65 th birthday of Professor
Barna A. Szabo



Conference Sponsored in Part by:

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Summaries of Papers

May 31 - June 2, 2000

Washington University, St. Louis, USA

This booklet contains the abstracts of all papers presented at pFEM2000. The abstracts are listed in alphabetical order of first author of each paper, started with the Plenary Talks, followed by the Parallel Talks. For reader's convenience, the abstract number is enclosed as well.

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International Journal for Numerical Methods in Engineering
and a special issue of the journal

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STATISTICAL ANALYSIS OF 3D MULTIPLE-SITE FATIGUE CRACK GROWTH IN AGING AIRCRAFT USING A *hp*-VERSION OF FEM

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The present paper deals with statistical analysis of fatigue crack growth and crack coalescence in riveted joints in real-life aircraft structures. The practical problem in mind is a part of the "Ageing Aircraft" problem, i.e. the case when old aircraft exhibit many small fatigue cracks which, in relatively few load cycles, link-up to form large cracks which jeopardize aircraft safety.

In the mathematical model, several simplifications are introduced what regards contact and friction between rivets/skin and skin/stiffeners. The equations of 3D elasticity are assumed valid, i.e. in standard notation,

$$(1) \quad \begin{aligned} C_{ijkl} u_{j,kl}(t) &= 0 \quad \text{on } \Omega \\ C_{ijkl} n_j u_{k,l}(t) &= T_i \cdot (1 + \sin(\omega t)) \quad \text{on } {}^N\Gamma(t, K) \\ u_i(t) &= 0 \quad \text{on } {}^D\Gamma \end{aligned}$$

where $u = (u_1, u_2, u_3)^T$ is the displacement vector, $T = (T_1, T_2, T_3)^T$ the traction vector, $n = (n_1, n_2, n_3)^T$ the normal vector, t is time, ω an angular frequency such that inertia effects become negligible. The domain Ω , with surface $\Gamma(t) = {}^N\Gamma(t) \cup {}^D\Gamma$, contains $K(t) \subset I$ cracks of sizes and shapes characterised by a few parameters $\{a_i(t), b_i(t), r_i(t) \mid 1 \leq i \leq K(t)\}$. Crack sizes increase with time due to the time varying load (fatigue). The domain of interest is that of a typical aircraft fuselage with skin, spars, frames, doublers and rivets, all parts modelled as full 3D objects in the numerical analysis. This leads, in the FE-analysis to algebraic systems of equations having $> 10^6$ degrees of freedom. In an experimental study it was found that the model used accurately predicts multiple-site fatigue crack growth in a butt joint, given the initial crack pattern.

A Monte Carlo simulation technique is used to calculate statistical distributions of the fatigue life for thousands of different initial crack flaw distributions in aircraft. The fatigue crack growth, crack coalescence, crack branching ("flapping") is followed deterministically, with control of error, for each of the initial damage scenario considered.

This type of analysis requires typically 10^6 - 10^7 accurate solutions of (1) for various boundaries ${}^N\Gamma(t, K)$ (crack sizes/distributions). A mathematical *splitting* method, which provide the high efficiency needed, is used. In the splitting scheme, the problem is divided into subproblems, which are solved using a *hp*-version of the finite element method, where the solution sought is obtained by proper superposition.

Convergence to exact solution for stress intensity factors is exponential in the pre-asymptotic range.

For statistical analysis on domains of present complexity, the computational effort might be very large. However, the mathematical splitting of the problem can be mapped very efficiently on a cluster of SMP-computers. A new solution paradigm is developed for *control* of the analysis which shows excellent *scalability*. On a 60 server cluster having in all 1000 CPu's, a computational speed > 0.5 TFLOPS was recorded. Finally, results from large-scale analysis of a large stiffened panel are presented and conclusions drawn.

THE p-VERSION OF FEM, FROM BIRTH TO MATURITY AND BEYOND

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The first part of the talk will briefly address the ideas behind the first phase of the development of the p-Version, especially with respect to the work of B. Szabó and the major results obtained later. The second part of the lecture will elaborate on the generalized finite element method (GFEM) which uses special shape functions related to the problem. This leads to the higher accuracy of the solution and its efficiency. GFEM can also be used to avoid the need of the mesh generator for very complex domains for example when the domain has many inclusions. GFEM is very flexible and utilizes all advantages of the standard h- and p- Version of FEM.

DISCONTINUITIES IN FINITE ELEMENTS AND MESHFREE METHODS AND LEVEL SETS

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This paper presents new methods for the construction of functions with arbitrary discontinuities and discontinuous derivatives. In finite element methods, these discontinuities are completely independent of the finite element mesh. In meshless methods, they have advantages over the visibility techniques developed in Belytschko, Lu and Gu [1]. The techniques are based on the concepts originated in Belytschko and Black [2] and Moen et al. [3]. The applications of the techniques are diverse: cracks and crack growth, shear bands, and phase changes are some of the areas of modeling which require discontinuities. Although evolving discontinuities are often treated by remeshing, this becomes quite awkward when the evolution of many discontinuities are considered.

Let the discontinuities in the dependent variable $u(\mathbf{x})$ occur on surfaces Γ_α , which are described by signed distance functions $f(\mathbf{x}) = \min_{\bar{\mathbf{x}} \in \Gamma_\alpha} \|\mathbf{x} - \bar{\mathbf{x}}\| \text{sign}(\mathbf{n} \cdot (\mathbf{x} - \bar{\mathbf{x}}))$, where $\bar{\mathbf{x}}$ is a point on the surface of discontinuity Γ_α and \mathbf{n} is a unit normal to the surface of discontinuity. The discontinuity surface is represented in the meshfree and finite element models by $f(\mathbf{x}) = f_I N_I(\mathbf{x})$, where repeated indices herein are summed over the appropriate range. The position of the discontinuity can be updated by level set methods. When the support is bisected, the approximation is given by

$$u(\mathbf{x}) = \sum N_I(\mathbf{x})(u_I + a_I \bar{H}(f_\alpha(\mathbf{x})))$$

where $\bar{H}(\mathbf{x})$ is a modified step function. The coefficient of a_I is called the enrichment. For a slit support, i.e. a support in which the discontinuity ends, the approximation is

$$u(\mathbf{x}) = \sum N_I(\mathbf{x})(u_I + a_{I\beta}(b_\beta(\mathbf{x}) - b_\beta(\mathbf{x}_I)))$$

where $b_\beta(\mathbf{x})$ are branch function around the discontinuity. Generally more than one branch function is needed for each enriched node; examples are

$$b_i(\mathbf{x}) = \left[r \sin \frac{\theta}{2}, r^2 \sin \frac{\theta}{2} \right] \text{ or } b_i(\mathbf{x}) = \left[\sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2} \sin \theta \right]$$

As can be seen in the branch functions on the left, they are discontinuous across the line $\theta = \pi$ but continuous and well behaved in the domain surrounding the discontinuity. For linear fracture mechanics, the branch functions on the right have been used in Belytschko and Black [2]. The last three functions are not discontinuous, but were added to speed the convergence to elastic fracture problems. Enrichments for discontinuities in the derivatives and for single components of a vector function can be constructed similarly. The latter can be used to model shear bands. Enrichments have also been developed for intersecting discontinuities, see [4].

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THIRTY YEARS OF THE p AND h-p FINITE ELEMENT METHODS - A HISTORICAL SURVEY

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As two major approaches of today's finite element method, the p and h-p versions have significantly developed in the past three decades. The development of the p and h-p finite element methods not only provides a reliable computational tool to modern engineering/scientific computing, but also offers fresh insights into many engineering and scientific problems. It has substantially enriched the knowledge of the finite element method and many relevant fields of mathematics such as approximation theory, regularity theory of partial differential equations with piecewise analytical data, and theory of shells and plates.

Originated from the computation of structural mechanical problems, the p and h-p finite element methods have made a splendid journey, from the birth in the later 1960's to today's state of the art. The p and h-p finite element methods are now widely used in almost every branch of engineering. Various large-scale commercial and research codes of the p and h-p version are available to engineering computation in industry as well as many research institutes and universities. These codes furnished with well-developed algorithms and combined with modern computer technology make it possible to solve engineering problems of great complexity. The p and h-p finite element methods have successfully penetrated into today's engineering practices and academic research.

The research on the theory of the p and h-p finite element method, started in the later 1970's, has established a strong mathematical foundation, which is able to sufficiently support the scientific and engineering computing and to offer new ways of improving the effectiveness of engineering computations.

The tremendous efforts and contributions from engineers and mathematicians lead to the great achievements of the p and h-p finite element methods in the twentieth century. In the year of 2000, it is worth while to address the history of the p and h-p finite element methods for better understanding in the future. The survey will give a review of major developments of the p and h-p finite element methods in past 30 years on the following aspects: approximation, algorithms, engineering applications, mathematical framework, and solvers of linear algebraic equations.

The survey will address some outstanding problems of p and h-p finite element methods in the beginning of new millennium.

UNSTRUCTURED SPECTRAL/hp ELEMENTS FOR TURBULENCE SIMULATIONS

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In the presentation we will discuss several versions of high-order bases (some inspired by Prof. Szabo's pioneering work) and present algorithms and results for simulation of turbulent flows in complex-geometry domains. In particular, new results on turbulent wakes at Reynolds number of 100,000 will be presented for first time.

WHY 2p IS NOT ENOUGH FOR Y2K AND BEYOND

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Today, we are truly in the midst of another Industrial Revolution: a digital revolution. And, just as in the case of the first, those organizations that can quickly attain mastery of the new paradigm will excel beyond their competition, and those that don't will quickly see that there cost, schedule and quality will not meet the new standards of the time. And, just in the case of the first, rapid deployment of new technology is met with significant technical and cultural challenges, often defeating most well intentioned efforts to provide new tools and integration solutions. This industrial revolution spans the entire techno-cultural foundation of our everyday life effecting everything from how we buy a book to how we develop advanced products in a fully expanded digital enterprise.

The year 2000 (Y2K) and beyond will mark the beginning of a new phase of the digital revolution. In the last several decades, software developers have put enormous capability into the hands of the industrial community for the development of products. Today, the challenge is how to integrate these software tools and, subsequently, what appears to be, a vast number of "islands of automation". In the new phase, the next step is to create an integrated, configuration controlled environment that allows the seamless flow of information from one discipline to the next. New methods that allow effective integration, information quality assurance, and increased functional performance are the new challenges.

Finite Element analysis is a key component of the product development process. Developments in FE methods have certainly provided enormous benefit to the analysis community. In recent times, of particular value is the higher order polynomial, p-version, finite element method. This method has a robust integration with the design geometry and its boundary conditions, provides an error estimate of the analysis, and provides functional methods that are not easily achieved in the tradition FE approach.

Effective and continuous evolution to enterprise wide integration and the enabling technologies requires a skill mix of enormous proportion. Tool benchmarking of the past needs to be replaced by integration projects that focus on culture and global integration. Integration solutions need to be enterprise wide, provide stable functionality, crafted for continuous improvement, and provide methods for an order of magnitude reduction in cost and schedule with significantly increased quality.

MULTISCALE HIGH ORDER FEM FOR PLATES AND SHELLS

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The efficient numerical modelling of laminated composites has received increasing attention in recent years – examples are sandwich plates and shells, fiber-reinforced composites and the like.

While global response to external loadings can be reliably assessed on the basis of averaged, or homogenized, models, interlaminar stresses can only be obtained by resolution of the small scales of the problem.

This requirement of SCALE RESOLUTION and the low solution regularity at the matrix-composite interface seems to conflict with the use of high order, spectral or hp-FE approaches which is commonly based on large elements with smooth shape functions of high order.

We present a new approach for the design of high order multiscale FEM for sandwich and lattice materials. The elements discretize concurrently macro- and microstructure of the material.

The FEM realize exponential convergence independently of the number of layers in the sandwich/ the number of fibers per unit length. It is based on a new spectral approach to homogenization and on nonpolynomial shape (or director) functions in the Finite Element Analysis.

Numerical results for composite material with periodic structure and for sandwich plates and shells will be presented. This is joint work with A.M. Matache (student), I. Babuška (Texas), B.A. Szabó and R. Actis (St. Louis, USA).

USE OF hp FINITE ELEMENT METHODS IN MATERIAL MODELING

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In this lecture, we discuss several difficulties in the implementation of hp and p finite element methods. We describe a number of techniques for overcoming these difficulties. The goal is to develop a convenient data structure, useful for developing robust application codes, which allows the implementation of mesh adaptivity with non-uniform distributions of polynomial order of p. These processes involve parallel nodal constraint algorithms introduced in the 80s [1], the use of PUMs as a basis of hp approximations, and recent implementations of discontinuous Galerkin methods.

Next, we focus on applications of hp methods to the analysis of highly heterogeneous materials. We demonstrate how parallel hp methods can be used as an analysis tool for adaptive modeling, as well as adaptive meshing. Error Estimations and are also presented.

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A REVIEW OF THE CORPORATE INPACT OF p-TECHNOLOGY PIONEERED BY PROF. SZABO

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In the early 1970's, Professor Barna Szabo pioneered an exciting new way to enhance the reliability of numerical simulation. This has given rise to a number of companies that were founded to commercialize and expand this technology in a number of different areas. This presentation will review these corporations and their contributions to the overall technology field.

ANALYSIS OF BONDED AND FASTENED REPAIRS BY THE P-VERSION OF THE FINITE ELEMENT METHOD

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The development of efficient and reliable methods for the analysis of fastened structural connections and bonded joints is among the most important problems in aerospace applications because these connections are common sites of failure initiation.

Realistic representations of fastened connections must account for the diameter of fastener holes, the stiffness of the fasteners, the effects of interference fitting or looseness of fit, contact between the fasteners and the connected plates, and yielding of the plates. Consequently the complexity of the models is such that it is difficult and time consuming to treat them using conventional finite element or boundary element procedures.

Similarly, the treatment of bonded joints is complicated by the fact that bonded patches are typically composed of several thin layers of orthotropic material and the adhesive material may yield under load, requiring a nonlinear analysis.

Fortunately, these problems lend themselves to standardization in FEA-based handbook libraries. Structural connections can be described in a parametric form. Therefore a topologically similar family of connections need to be meshed only once. The FEA-based handbook library provides for the archival and recall of entries for analysis. Users need only enter the parameters. The finite element mesh, being associative, is adjusted automatically. Therefore users need not be concerned with meshing or other details of the analysis process.

The finite element solution and the computation of all data of interest are performed automatically and the results are displayed in tabular and graphical forms. Automatic quality control tests are performed and reported for purposes of documentation. There are provisions for the computation of margins of safety. The advanced FEA-based handbook library produces results of sufficient reliability to permit certification of repairs and joints on the basis of computed information. This will be demonstrated by examples.

A POSTERIORI ERROR ESTIMATION FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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The discontinuous Galerkin method (DGM) is an appealing approach to address problems having discontinuities, such as those that arise in hyperbolic conservation laws. Originally developed for neutron transport problems, the DGM has been used to solve ordinary differential equations and hyperbolic, parabolic, and elliptic partial differential equations. The DGM may be regarded as a way of extending finite volume methods to arbitrarily high orders of accuracy. The basis is piecewise continuous relative to a structured or unstructured mesh. As such, it can sharply capture discontinuities in the solution. Regardless of order, the DGM has a simple communication pattern to elements with a common face that makes it useful for parallel computation. It can handle problems in complex geometries to high order. In order for the DGM to be useful in an adaptive setting with h -refinement (mesh refinement and coarsening) or p -refinement (method order variation), techniques for estimating the discretization errors should be available both to guide adaptive enrichment and to provide a stopping criteria for the solution process. We will show that the DGM discretization finite element solution of one-dimensional hyperbolic conservation laws with degree p exhibit superconvergence at Radau points of degree $p + 1$ with the fixed endpoint selected at the upwind end of each element. We use this superconvergence result to construct asymptotically exact a posteriori error estimates for hyperbolic systems of partial differential equations. Finally, we present numerical results for several computational examples that show the efficiency of our a posteriori error estimator.

A-52

SOME REMARKS ON A POSTERIORI ERROR ESTIMATION FOR PARAMETER DEPENDENT PROBLEMS

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Standard a posteriori error estimators when applied to problems containing a parameter are often found to be sensitive to the parameter in various limiting cases. This either means that as the parameter approaches certain limits, the estimator will either become increasingly pessimistic or increasingly unreliable. The talk will present examples of this type of behaviour and discuss methods whereby the robustness and reliability of the estimators may be restored.

A-53

THE STABILITY OF HIGH ORDER MIXED FINITE ELEMENT FAMILIES FOR INCOMPRESSIBLE FLOW

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The talk will discuss the stability of mixed finite element families on meshes containing high aspect ratio affine quadrilateral elements. Sharp theoretical estimates for the dependence of the inf-sup constant on the aspect ratio, the element size and the spectral order of the elements will be presented, along with supporting numerical evidence.

***p*-FEM FOR FORMULATING AN ELASTIC CRITERION
FOR PREDICTING MECHANICAL FAILURES AT 2-D
SINGULAR POINTS**

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Failures of mechanical components and electronic devices usually start at reentrant corners or the intersection of multi-material interfaces. To date the structural engineering community is lacking a criterion for predicting the failure onset in the vicinity of these singular points. In linear elastic fracture mechanics for brittle materials it is well known that a single parameter of the solution in the vicinity of the crack tip (called stress intensity factor- K_c) can be correlated to a material dependent value for predicting fracture onset. Same methodology may be applicable for singularities associated with sharp re-entrant corners (V notch tip), only that in this case the situation is much more complicated due to many more variables involved in the expression of the singular elastic solution [1]. Recently, a general method for the accurate and reliable computation of all the information associated with the singular solution (generalized stress intensity factors - GSIFs - and associated eigen-pairs) in the vicinity of any 2-D elastic singular point has become available in a p-FEM code [2]. This information is used herein for investigating possible assumptions on failure criteria from V-notch tips. The correlation of a functional based on the computed GSIFs and associated eigen-pairs to experimental observations may be the key for the formulation of a failure criterion at singular points. Herein we address the computation of the required information at singular points using the p-FEM, the need for a functional instead of a single value in the vicinity of the singularity, and preliminary correlations to experimental observations. The elastic strain energy density in the vicinity of the V-notch tip is suggested as the failure driven functional. Brief explanation is provided. Let us consider the vicinity of a re-entrant corner as shown in Figure 1.

The elastic strain energy $\tilde{U}(\bar{u})$ density in Ω (the vicinity of the notch tip) is:

$$\frac{U(\bar{u})}{A_n} = \frac{1}{2} \frac{B(\bar{u}, \bar{u})}{\pi R^2 \left(1 - \frac{\omega}{2\pi}\right)} = \frac{1}{2} \iint_{\Omega} ([D]\bar{u})^T [E] ([D]\bar{u})_z d\Omega / \pi R^2 \left(1 - \frac{\omega}{2\pi}\right) \quad (1.1)$$

where $[D]$ is a differential matrix, $[E]$ is the material matrix and t_z is the thickness [3]. Based on the weak formulation, instead of $B(\bar{u}, \bar{u})$ one can compute the linear form $F(\bar{u})$ (involves only a 1-D integration along the circular path Γ_R):

$$U(\bar{u}) = \frac{1}{2} B(\bar{u}, \bar{u}) = \frac{1}{2} F(\bar{u}) = \frac{R}{2} \int_{\theta_0}^{\theta_1} [\bar{T}^T \cdot \bar{u}]_{z=R} d\theta \quad (1.2)$$

where \bar{T} denotes the traction vector. Let us denote $[\bar{T}^T \cdot \bar{u}]_{r=R} d\theta$ as "expr-1".

In the vicinity of the singular point, the asymptotic expansion of the displacement field is:

$$\bar{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \sum A_i r^{\alpha_i} \begin{Bmatrix} u_x^{(i)}(\theta) \\ u_y^{(i)}(\theta) \end{Bmatrix} \quad (1.3)$$

Where $\alpha_i, f_i(\theta)$ are called eigen-pairs and A_i are the GSIFs.

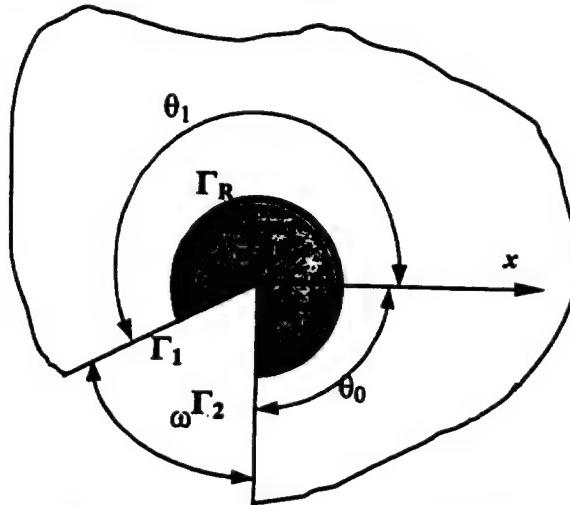


Figure 1 - Singular point vicinity: Notations.

Obtaining the stress tensor from (1.3):

$$\bar{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \sum A_i r^{\alpha_i-1} \begin{Bmatrix} \sigma_{xx}^{(i)}(\theta) \\ \sigma_{yy}^{(i)}(\theta) \\ \sigma_{xy}^{(i)}(\theta) \end{Bmatrix} \quad (1.4)$$

the strain energy density is completely determined from the GSIFs and associated eigen-pairs once obtained from p-FEM results:

$$\begin{aligned} \tilde{U} = & \left(\frac{1}{(2\pi - \omega)} \right) \sum A_i^2 R^{2(\lambda_i-1)} \\ & * \int_{\theta_0}^{\theta_1} \left[(\sigma_{xx}^{(i)}(\theta) \cos(\theta) + \sigma_{xy}^{(i)}(\theta) \sin(\theta)) u_x^{(i)}(\theta) \right] + \left[(\sigma_{xy}^{(i)}(\theta) \cos(\theta) + \sigma_{yy}^{(i)}(\theta) \sin(\theta)) u_y^{(i)}(\theta) \right] d\theta \end{aligned} \quad (1.5)$$

Since in the vicinity of the singularity, for $R \ll 1$, the first term ($i=1$) in (1.5) dominates (λ_i become larger as i increase), thus an excellent approximation for \tilde{U} is obtained by considering the first term alone. Examining (1.5) it may be noticed that the strain energy density is a measure depending both on the GSIFs and the eigen-pairs, which depend not only on the loading but on the re-entrant corner angle as well.

The validity of the strain energy density as a criterion for failure initiation is verified by comparing to experimental observations reported in [4].

p-FEM analyses were performed for three and four point bending re-entrant V-notched PMMA specimens shown in Figure 2, as well as sensitivity tests in order to establish the influence of the load and support representation on the results in the vicinity of the singular point, and then compared our results to the experiments done by Dunn et al. [4].

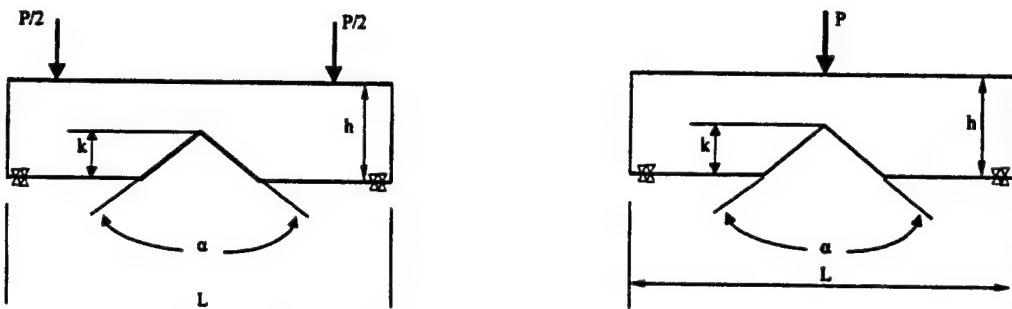


Figure 2 - Four point and three point specimens

This talk will provide the preliminary results of our analysis and their correlation with experimental observations. If time allows, the influence of a rounded V-notch tip on the sensitivity of the results will be addressed.

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ON THE RELEVANCE OF THE *inf-sup* CONDITION IN SPECTRAL PROJECTION METHODS

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The recent literature on variational projection methods is permeated by the idea that they can be used with spatial interpolations not satisfying the *inf-sup* condition, often referred to as the LBB condition from Ladyzhenskaya, Babuška and Brezzi. In fact, this kind of methods are often implemented using equal order interpolations for velocity and pressure and employed as stabilization techniques for the solution of the incompressible Navier-Stokes equations. However, severe spatial instabilities are found to plague the numerical solutions obtained by Galerkin finite element projection methods, as thoroughly illustrated in [3].

Since very efficient projection methods based on a Galerkin-Legendre spectral approximation have been developed in the last years [6], it is worthwhile to clarify whether the aforementioned stability condition on the spatial representation of velocity and pressure fields has sensible consequences also in the spectral case.

In the present paper we show that only by using interpolations of different order such as P_N-P_{N-2} for velocity and pressure [3] stable projection method of a spectral type can be obtained for computing incompressible flows in 2D.

In particular, the incremental version of the projection method proposed in [2] requires to solve two uncoupled problems: first solve the viscous step problem

$$(2) \quad \begin{cases} \frac{\mathbf{u}^{k+1}-\mathbf{u}^k}{\Delta t} - \nu \nabla^2 \mathbf{u}^{k+1} = \mathbf{f}^{k+1} - \nabla(2p^k - p^{k-1}) - (\mathbf{u}^k \nabla) \mathbf{u}^k, \\ \mathbf{u}^{k+1}|_{\partial\Omega} = \mathbf{b}^{k+1}; \end{cases}$$

then solve the incompressible projection step problem formulated as a Neumann boundary value problem for the pressure increment

$$(3) \quad \begin{cases} -\widehat{\nabla}^2(p^{k+1} - p^k) = -(\Delta t)^{-1} \nabla \dot{\mathbf{u}}^{k+1}, \\ \frac{\partial(p^{k+1} - p^k)}{\partial n}|_{\partial\Omega} = 0. \end{cases}$$

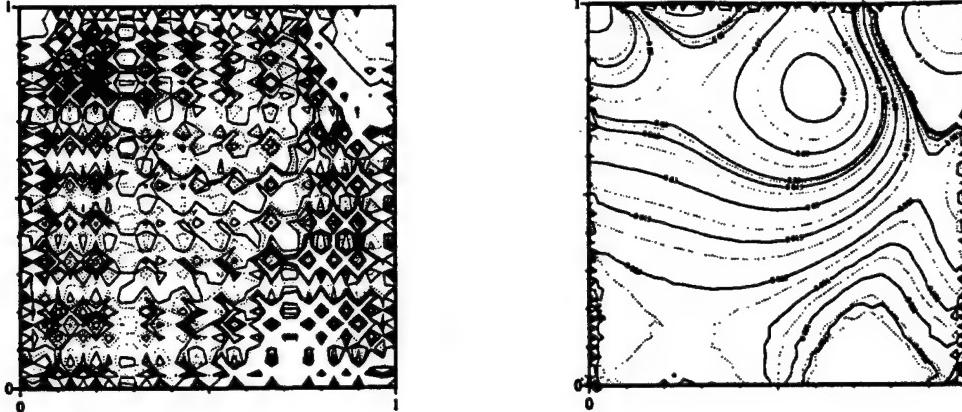


Figure 1: Unstable (left) and stable (right) steady pressure fields for $Re = 100$ obtained by an equal order approximation $P_{40}-P_{40}$ and a mixed method $P_{40}-P_{38}$, respectively.

The elliptic equations for velocity and pressure have been recast in a weak variational form by the Galerkin-Legendre spectral method of Shen [5]. The solution of the fully discrete elliptic equations are calculated by a direct method with eigen-decomposition in both spatial directions. The nonlinear terms are evaluated by standard pseudospectral technique based on Gauss-Legendre quadrature points.

Figure 1 contains spectral solutions of the standard driven cavity problem at $Re = 100$ obtained by means of the non-incremental method with $\Delta t = 10^{-4}$. The highly oscillatory pressure field obtained by equal order spatial discretization P_N-P_N is compared with the pressure field computed by a compatible mixed approximation P_N-P_{N-2} . The limited oscillations in the latter, located near the boundary, are due to the Gibb's phenomenon induced by the singularity of the pressure field in the upper corners. These residual oscillations disappear completely by subtracting the singular component of the solution as has been found in other computations to be presented in the final paper.

It is to be remarked that such an instability has been observed for the non-incremental method, provided the time step is sufficiently small, as well as for the incremental method previously described. Moreover, the same unstable results have been obtained also when the Poisson equation for the pressure has been solved by enforcing the Neumann boundary condition in an essential way, as proposed by Shen in [5, page 1500] and in a private communication, instead of the classical weak variational form.

The results to be presented confirm that, in compliance with the LBB condition, the incremental (pressure correction) method must be implemented by means of interpolations of different order to provide a stable spectral scheme for any time step. Such a method gives nonoscillatory velocity and pressure fields under ordinary circumstances, without requiring to introduce any *ad hoc* stabilization technique.

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INCOMPRESSIBLE NAVIER-STOKES SOLUTIONS BY A TRIANGULAR SPECTRAL/*hp* ELEMENT PROJECTION METHOD

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The projection method dates to more than thirty years ago and it has been intensively used for the numerical simulation of incompressible viscous flows by means of Navier-Stokes equations [1, 2].

Notwithstanding, a completely satisfactory mathematical theory for this method in its incremental version has been exploited only recently by J.-L. Guermond and L. Quartapelle [3], using a proper variational formulation.

In this theory the projection method is characterized by the presence of two different approximation spaces for the intermediate and final velocity, the latter belonging to $H^{\text{div}}(\Omega)$. Such a velocity can be eliminated from the implemented algorithm leading to a very simple and efficient procedure, as shown in [4]. Moreover, the incremental method is characterized by a time splitting error that allows to develop a truly second order accurate second order integration scheme.

In the field of high order methods, quite recently, a new, well conditioned, basis on triangles has been proposed by Dubiner [5] and employed by Sherwin and Karniadakis [6, 7] to implement spectral/*hp* element method on triangles and tetrahedra. The discretization has been applied to the Navier-Stokes equations, employing the high order splitting scheme proposed in [8].

The aim of the present work is to report the first application of the projection method proposed in [4, 3], in the context of a spectral/*hp* element spatial discretization technique. In the proposed method, differently from the original one, the convection term is advanced explicitly, the viscous term being advanced implicitly. An efficient method, suitable for the Direct Numerical Simulation of two-dimensional incompressible viscous flows in two dimension, is obtained. The spatial discretization is carried out by a mixed element formulation, based on the spectral/*hp* element

method proposed in [6], employing different piece-wise polynomial spaces for velocity and pressure in order to satisfy the *inf-sup* compatibility condition. This allows to extend the Taylor–Hood finite elements to higher degrees as in [9].

The stability and approximation properties of the proposed method are assessed by two test cases. The first one, characterized by a solution in closed form, is meant to prove the spectral accuracy of the method (see Table 1). The second one exemplifies the capability of the proposed method to cope with realistic geometries.

| Degree | u_x | u_y |
|--------|-----------------------|-----------------------|
| 3 | $2.40 \cdot 10^{-3}$ | $1.42 \cdot 10^{-3}$ |
| 5 | $3.07 \cdot 10^{-5}$ | $1.90 \cdot 10^{-5}$ |
| 7 | $2.89 \cdot 10^{-7}$ | $1.69 \cdot 10^{-7}$ |
| 9 | $1.84 \cdot 10^{-9}$ | $1.01 \cdot 10^{-9}$ |
| 11 | $8.32 \cdot 10^{-12}$ | $4.34 \cdot 10^{-12}$ |

Table 1: $L^2(\Omega)$ error as a function of the polynomial degree.

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HELMHOLTZ-NEUMANN SPECTRAL SOLVER WITH AN ESSENTIAL TREATMENT OF BOUNDARY CONDITIONS

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The prototype algorithm for solving 2D Poisson and Helmholtz equations in a rectangle by spectral methods is the diagonalization technique proposed in [5, p. 150], see also [4, p. 133] and the reference therein. This algorithm has been implemented by Haidvogel and Zang using Chebyshev polynomials and the tau method [6]. Recently, the advantages of resorting to the standard Galerkin formulation in approximating elliptic problems by means of high order polynomials have been pointed out [7] and a thorough error analysis of Legendre spectral methods for the Poisson equation and the Stokes problem has been carried out in [3].

In particular, a Legendre basis $P_{0,N}$ has been proposed by Jie Shen [8], leading to a pentadiagonal mass discrete operator and to a stiffness operator equal to the identity matrix. A direct Helmholtz-Dirichlet solver based on variable separation and implementing a diagonalization solution technique has been derived with possibly nonhomogenous boundary conditions imposed by an analytical lifting.

The present authors [1] proposed an implementation of the aforementioned basis to build a direct solver using a double diagonalization technique and a discrete lifting of the Dirichlet boundary data. The discrete lifting is carried out by suitably augmenting Shen's $P_{0,N}$ basis—a technique proved successful also for the 3D problem [2]. In principle, the augmented basis should be able to accommodate general boundary conditions including the pure Neumann problem. Unfortunately, in such an extension the tensor product structure of the solver can not be preserved owing to the particular form assumed by the stiffness one-dimensional operator in the presence of derivative boundary conditions.

In the present work, a Helmholtz-Neumann direct solver fully implementing the tensor product form is proposed. Following a suggestion by Shen, we impose the Neumann boundary conditions in an *essential* way to develop a 2D spectral solver based on variable separation and double diagonalization technique.

The efficiency and accuracy of the proposed numerical solver will be illustrated by some numerical tests and by its application in the context of a Galerkin-Legendre spectral projection method for the Navier-Stokes equations.

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TIME SPLITTING/SPECTRAL ELEMENT METHOD FOR NAVIER-STOKES EQUATIONS

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This contribution is a presentation of some time splitting methods applied to the incompressible Navier-Stokes equations. We describe among them the most currently used: Chorin-Temam scheme [1], Kim and Moin scheme [2] and Goda scheme [3] in their first and second order versions. The spacial discretization is carried out using the spectral element method for the prediction/diffusion and for the correction/pressure-continuity steps. We discuss the advantages and drawbacks of each of them, with a particular focuss on the artificial boundary condition on the pressure which would be responsible of a numerical boundary layer.

We recall the mathematical results already proven, we emphasize on the work of Ralph Rannacher [4], of Jie Shen [5] and of Jean Luc Guermond [6]. Then, we show numerical results for analytical solutions so that we can compare time splitting algorithms with the Uzawa [7] solution, so as to highlight the reliability of this kind of algorithms. Moreover, numerical simulation of Navier-Stokes flows will be given for different methods proving again the efficiency of these algorithms.

Finally, we present some interesting variants of each projection method that is well suited to parallel computations and provide numerical evidencies of its efficiency so as the substantial gain in the CPU time.

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HIGHER ORDER MODELING OF CONTINUA BY FINITE ELEMENT, BOUNDARY ELEMENT, MESHLESS, AND WAVELET METHODS

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This paper is an overview of the higher order functional modeling schemes based on finite element, boundary element, meshless, and wavelet methods. It is based on the research work of the principal author and his students over a period of 25 years, primarily, related to one-D and two-D problems. The first discrete numerical method is well established, so discussions are mainly focused on the last three methods. Apart from critically discussing the main theoretical, algorithmic, and implementation characteristics of the methods, the paper expounds on the relative merits and demerits of the methods. The results of a number of smooth and non-smooth example problems are presented. Finally, the future potential of the wavelet method in discrete numerical modeling of continua is explored.

THE hp -MORTAR FINITE ELEMENT METHOD FOR MIXED ELASTICITY AND STOKES PROBLEMS

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After the work done by P. Seshaiyer and M. Suri, [1] on the hp -mortar finite elements, the next step is to extend this method to the nearly incompressible elasticity model formulated as a mixed displacement-pressure problem so as to Stokes equations in primal velocity-pressure variables. Within each subdomain the local approximation is designed on div-stable hp -mixed finite elements. The displacement is computed in a mortared space while the pressure is not subjected to any constraints across the interfaces. By a Boland-Nicolaides argument we prove that the discrete saddle point problem satisfies a Babuška-Brezzi *inf-sup* condition. The *inf-sup* constant is optimal in the sense that it depends only on the local (to the subdomains) characteristics of the mixed finite elements and, in particular, it does not increase with the total number of the subdomains. The consequences, that we are aware of, of such an important result are twofold.

1. The numerical analysis of the approximability properties of the hp -mortar discretization for the mixed elasticity problem allows to derive an asymptotic rate of convergence that is optimal up to $\sqrt{\log p}$ in the displacement.
2. When the mortar discrete problem is inverted by substructured iterative methods based on Krylov subspaces with block preconditioners, resuming the proofs of L. Pavarino and O. Widlund, [2] completed with those of Y. Achdou, Y. Maday and O. Widlund, [3] the condition number of the solver depends polylogarithmically on (p, h) , on the local *inf-sup* constants and does not on the total number of the subdomains.

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DUAL-MIXED p AND hp FINITE ELEMENTS FOR ELASTICITY PROBLEMS

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Application of displacement-based h elements for constraint problems of elasticity and plasticity often results in zero or nearly-zero rates of convergence in the displacements, in energy norm and, more significantly, in the stresses. Well-known examples are the incompressibility constraint, the shear and membrane constraints appearing in plate and shell models when the thickness of the plate or shell approaches to zero, and the constraint resulted by the boundary layer phenomenon in plate and shell models.

Improvement of displacement-based h elements in linear analysis seems to have a tradition; the applicability of those numerical 'tricks' (reduced-selective integration, hour-glass stabilization) in nonlinear analysis is nevertheless questionable. Higher-order displacement-based p elements are proved to be robust in the displacement approximations, the computed stresses, however, are not free of locking when problems of incompressible elasticity are considered [1].

The use of mixed p and hp finite elements is one of the possible solutions to overcome the aforementioned numerical difficulties. A significant advantage of mixed finite element models over displacement ones is that they are robust with respect to both h and p extensions, which is particularly interesting in applications requiring not only efficient high-, but locking-free low-order elements as well [2].

Although dual-mixed variational principles provide the possibility of approximating the stress space directly (the variable of primary interest in many engineering applications), creating stable finite element spaces using *a priori* symmetric stresses is not a simple task [3]. This is probably one of the reasons for an increasing interest in dual-mixed variational principles using non-symmetric stresses in finite element analysis. Development of higher-order dual-mixed p elements with curved boundaries seems, however, to be rather difficult when the stress space is approximated directly (even if it is not assumed to be *a priori* symmetric). The main problem is the assurance of inter-element surface traction continuity for curved elements, because the mapping function between the master and actual element is nonlinear.

This paper presents dual-mixed p and hp finite element models for two-dimensional elasticity problems. The variational formulation is based on Fraeijs de Veubeke's dual-mixed principle in terms of non-symmetric stresses and rotations [4]. *A priori*

satisfaction of translational equilibrium is achieved by introducing first-order stress functions which are the directly approximated variables (instead of stresses). This leads to a weak form which is analogous to that of the displacement-pressure formulation of elasticity and the velocity-pressure formulation of Stokes flow, allowing the use of the stable mixed spaces developed in [2]. Another advantage in approximating first-order stress functions directly is that their C^0 continuity across an element interface ensures the continuity of the surface tractions as well. This fact makes it possible to construct efficient dual-mixed elements with curved boundaries which has basic importance regarding p -extensions.

Numerical performance of three quadrilateral dual-mixed hp finite elements is demonstrated through simple examples. Focusing on nearly incompressible elasticity, numerical results are compared to solutions obtained by standard displacement-based hp finite elements. It will be shown that convergence of the energy norm as well as the stresses is independent of the incompressibility constraint when the present dual-mixed elements are applied, i.e. they are free of locking for both h - and p -extensions.

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THE SOLUTION OF A NONLINEAR EIGENVALUE PROBLEM USING P-VERSION OF FEM

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A number of physical phenomena are described by the Poisson equation

$$-\Delta u = f(x, y, u) \text{ on } \Omega.$$

This equation is a special case of the problem of the thermally orthotropic material. The two dimensional heat conduction equation is

$$-\frac{\partial}{\partial x} \left(k_x(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y},) \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial y} \left(k_y(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y},) \frac{\partial}{\partial y} \right) = f(x, y, u) \text{ on } \Omega \in \mathbf{R}^2, \quad (1)$$

where k_x and k_y are heat conduction coefficients along the orthotropic axes x and y , respectively.

A special case of the equation (1) is the nonlinear partial differential equation

$$-\frac{\partial}{\partial x} \left(\left| \frac{\partial u}{\partial x} \right|^{n-1} \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left(\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right) = f(x, y, u) \text{ on } \Omega, \quad (2)$$

where $0 < n < \infty$ real number. If $n = 1$ the operator at the left side of the equation (2) is reduced to $-\Delta u$.

We shall consider the eigenvalue problem of this type of equation

$$\frac{\partial}{\partial x} \left(\left| \frac{\partial u}{\partial x} \right|^{n-1} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right) + \lambda |u|^{n-1} u = 0 \text{ in } \Omega, \quad (3)$$

where Ω is a convex domain in \mathbf{R}^2 , under Dirichlet boundary condition

$$u|_{\partial\Omega} = 0.$$

The weak formulation of the problem of (3) is

$$\int_{\Omega} \left(|u_x|^{n-1} u_x v_x + |u_y|^{n-1} u_y v_y \right) dx = \lambda(n) \int_{\Omega} |u|^{n-1} u v dx$$

for any $v \in W_0^{1,n+1}(\Omega)$. It is known that (3) has a sequence of weak solutions

$(\lambda_k(n), u_k(n))$ in $\mathbb{R} \times W_0^{1,n+1}(\Omega)$, where $0 < \lambda_k(n) < \lambda_{k+1}(n)$, $k = 1, 2, \dots$ [2].

The first eigenvalue can be defined by the variational principle

$$\lambda_1(n) = \inf_{u \in W_0^{1,n+1}(\Omega)} \frac{\int_{\Omega} (|u_x|^{n+1} + |u_y|^{n+1}) dx}{\int_{\Omega} |u|^{n+1} dx}$$

The unique solution $(\lambda_1(n), u_1(n))$ of (3) is called the first eigenpair of (3).

We shall use the finite element method to approximate the first eigenvalue and the corresponding eigenfunction to the nonlinear eigenvalue problem of (3) [3]. The finite element discretization of the domain Ω is made by h and p version [1]. For the iteration of the generalized Rayleigh-quotient the Newton-Raphson method is applied such that the parameter n is increased step-by-step.

We have got approximate solutions for the radially symmetric solutions of the eigenvalue problem (3) when Ω is bounded by the curve

$$|x|^{\frac{1}{n+1}} + |y|^{\frac{1}{n+1}} = 1.$$

This curve plays the same role in case of the nonlinear differential equation (3) as the unit circle in the linear case when $n = 1$. The approximate solutions of the first eigenvalues obtained by Runga-Kutta method will be compared with the finite element solutions.

When Ω is bounded by a square the analytic solutions are known for the nonlinear eigenvalue problem. The numerical results obtained by the finite element method will be compared with the exact values. Illustrating the performance of the h and p version we represent the relative error of the first eigenvalue for different values of n . The superiority of the p -version will be demonstrated.

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TWO-DIMENSIONAL INTERFACE PROBLEMS

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Let Ω be a polygonal domain in \mathbb{R}^2 which is divided into nonoverlapping polygonal subdomains $\Omega_1, \dots, \Omega_J$ such that $\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j$.

Consider the interface problem of finding $u \in H_0^1(\Omega)$ such that

$$(*) \quad \sum_{j=1}^J \int_{\Omega_j} \rho_j \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H_0^1(\Omega),$$

where ρ_1, \dots, ρ_J are positive constants. It is well known that in the neighborhood of a cross point the solution u is not smooth even if f is a C^∞ function. In particular, the best that can be said about the elliptic regularity of $(*)$ is that there exists a number δ between 0 and 1 such that $f \in H^{-1+\delta}(\Omega)$ implies that $u \in H^{1+\delta}(\Omega) \cap H_0^1(\Omega)$. Depending on the coefficients ρ_j and the angles among the interfaces at the cross points, the number δ can be arbitrarily close to 0. This lack of regularity is responsible for the failure of simple finite element methods for interface problems.

In this talk we will present optimal order multigrid methods that take advantage of the singular function representations of u at the cross points and the extraction formulas which express the stress intensity factors in terms of u . These methods produce approximate solutions of the form $u_h = \sum \kappa_{l,h} s_l + w_h$, where the s_l 's are the singular functions and w_h is a P_1 finite element function on a (quasi) uniform grid with mesh size h . The convergence of u_h in the energy norm is of order $O(h)$ when $f \in L_2(\Omega)$, and $O(h^2)$ when $f|_{\Omega_j} \in H^1(\Omega_j)$ for $1 \leq j \leq J$. The numbers $\kappa_{l,h}$ also approximate the stress intensity factors at the rate of $O(h^{1+\delta})$ when $f \in L_2(\Omega)$, and $O(h^2)$ when $f|_{\Omega_j} \in H^1(\Omega_j)$ for $1 \leq j \leq J$.

**HIGH ORDER MIXED RWG BASIS FUNCTIONS FOR
ELECTROMAGNETIC SCATTERING IN
MULTILAYERED MEDIA AND APPLICATIONS**

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We will discuss two numerical issues in the electromagnetic scattering in multilayered media using mixed potential integral equation approaches. The first is the construction of high order RWG basis functions for curved surfaces and the second the fast calculation algorithm for the dyadic Green's functions for multilayered media. Basis functions for combinations of curved triangular and quadrilateral patches can be used. The Green's function calculation algorithm is applicable to arbitrary number of layers and frequency range. Several applications of electromagnetic scattering in a multilayered medium, such as three dimensional discontinuities in VLSI design and multilayered RF components and multilayered antenna, are provided.

**POINTWISE AND LOCAL ERROR ESTIMATES FOR
THE QUANTITIES OF INTEREST IN TWO
DIMENSIONAL ELASTICITY**

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Procedures for bounding the error in stress in a finite element solution have been under development by the current authors for over a decade. Recently the procedures have been refined to allow recovery of point-wise error bounds on curved boundaries and the computations have become inexpensive. Other researchers have relaxed the requirement that the error be bounded and have developed very successful error measures robust enough to guide adaptive finite element schemes. Reliable information about the exact solution follows after a sequence of solutions has been executed. We have taken a more pragmatic approach of trying to demonstrate that the mathematics associated with the finite element method is capable of providing a guaranteed bound on the error in stress at any user defined location in the solution domain. Reliable information about accuracy of the solution is then available following a single analysis. The 'error' referred to here is the discretization error associated with the design of the finite element mesh. These error measures can be used to guide refinement but also to terminate the analysis after an initial solution.

Procedures for bounding the error in local point-wise derivatives in a purely Neumann problem have been published. Current research is focusing on arbitrary Dirichlet and mixed boundary conditions and a nonlinear problem. The procedures require the recovery of a post-processed solution which smoothes the discontinuity in derivatives across element boundaries. Control of the error bounds at points close to or on the boundary relies on the use of the fundamental solution on a half plane for elasticity. The bounds on the error in the post-processed solution are guaranteed.

In this paper, we apply the procedures to linear two-dimensional elasticity with a stress concentration. Accurate bounds for the peak stress, corresponding to a tight envelope in which the exact solution lies, will be presented. A new procedure to solve patched residual based problems, to locally estimate the error, are being investigated and the results will be reported at the conference.

CW - 3

INTERFACING H-ELEMENT AND P-ELEMENT MODELS USING STATIC CONDENSATION

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A method for connecting a p-element model to an h-element model using static condensation is presented. This approach is based on equating strain energies at the connection interface of the two models. Using least square fit on the displacement fields at the interface, an equation is obtained which can be used to transform the stiffness matrix of the p-element model condensed to the interface to an "equivalent" stiffness matrix on the h-side of the interface. Similarly, loads acting on the p-element model can be condensed and transformed to the h-side.

This method is useful in cases where a component of a system demands more detailed finite element analysis than the rest of the system to obtain an accurate solution. Using this approach, this system can be broken down into two models: the component and the system without the component. The first model (the component) is analyzed using p elements to obtain accurate condensed stiffness and load matrices at the connection interface. These condensed matrices can then be used as superelements in subsequent h-element analyses on the second model.

This method is implemented as a new functionality in an upcoming release of Pro/Mechanica. Numerical examples based on this implementation will be provided using Pro/Mechanica and Nastran.

PARALLEL GUARANTEED-QUALITY h-REFINEMENT AND MESH GENERATION*

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In this paper we describe two basic problems associate to the introduction of concurrency in h-refinement for Delaunay meshes. Specifically, we address problems related to the correctness and the efficiency of parallelized 3D adaptive mesh generators based on existing scalar guaranteed quality Delaunay tetrahedralization algorithms.

Delaunay triangulation algorithms have been used very successfully for unstructured grid generation on sequential machines. Similarly, the Bowyer-Watson (B-W) kernel has been used successfully by Chew, Ruppert and many others to generate guaranteed quality meshes for 2D domains, and by Shewchuk for piecewise linear complexes in 3D —under certain constraints. Variations of this kernel have been used by many others; the main difference between the various algorithms is in the treatment of the domain boundaries (constraints). Traditionally, the B-W kernel is parallelized using domain decomposition; the challenge then is to maintain the Delaunay property across the boundaries of the subdomains. The main step in the B-W kernel, cavity expansion, is based on a breadth-first search over the mesh's data structures; sequentially, this is a very simple task to accomplish. In parallel, however, cavity construction becomes much more difficult, since cavities may extend across the boundaries (interfaces) separating adjacent regions of the mesh.

The expansion of these multi-region (MR) cavities can be synchronous or asynchronous, either halting or allowing the creation of new cavities in regions participating in the cavity expansion. In an asynchronous implementation, it is possible for MR cavities to intersect (share tetrahedra or faces with) local cavities or other MR cavities, which can result in a non-Delaunay re-tetrahedralization of the intersecting cavities. We analyze the possible intersection cases in a multithreaded program execution model and study the impact of concurrency on (1) the correctness of the parallel h-refinement algorithm, (2) the quality of the triangulation, and (3) "setbacks" in the progress of h-refinement parallel algorithms.

In addition to the difficulties of cavities which extend over multiple regions, problems with correctly updating a region near by external boundary in the face of concurrency also arise. For example, in a straightforward asynchronous parallelization of Shewchuk's conforming constrained Delaunay tetrahedralization algorithm, one

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must do away with the ordering on the splitting of edges, faces, and elements called for by the original algorithm. Not only is mesh quality affected by this change, but neither the correctness of the resulting mesh, nor the termination of the algorithm, can be proved. These problems exemplify the need for the creation of a wholly new parallel algorithm for Delaunay mesh generation.

While the parallelization of an existing algorithm seems appropriate at first glance, our experience shows that there is a need for a purely parallel algorithm. This new algorithm will be developed keeping in mind the problems, the inconsistencies, and the inefficiencies that arise due to concurrency. The complexity and the performance of our current parallel implementation clearly demonstrate the need for such an algorithm. Our results also show that parallel Delaunay meshing based on the B-W kernel is a foreseeable goal, although much work remains to be done to realize a usable and efficient guaranteed quality parallel mesh generator.

ERROR ESTIMATION, AUTOMATIC p- AND hp-ADAPTATION USING HIERARCHICAL CONCEPT

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This paper presents a summary of the authors Ph.D. Thesis [1]. A p-code named PHAME has been developed at the LTAS laboratory. This finite element program is able to solve static and dynamic problems in linear elasticity both for 2-D and 3-D applications.

A posteriori error estimation techniques can be classified in three families: 1) error estimation based on residual of equilibrium and stress discontinuities; 2) error in the constitutive relation; 3) error estimation based on gradient recovery. Excepting the equilibrated residual method [2], all other methods don't seem to be applicable to high degree elements. In most of the commercial p-codes, there isn't any estimator implemented and the solution is considered as good when the variations of some user defined values between two hierarchical solutions are small enough. In this paper, a new approach based on the convergence of hierarchical solutions is going to be presented.

In this work we've tried to take maximal benefit of the hierarchical formulation to estimate the discretization error. The main idea is that a solution is good when the structural response don't change too much when the discretization is improved. Our new approach based on engineering practice is quite reliable. A measure of the global and local quality of a solution can be obtained by comparing two hierarchical solutions; the error of p-1 solution can be estimated by using degree p solution as reference solution [4]. This leads to the following error indicator:

$$\varepsilon = \int_{\Omega} (\sigma_p - \sigma_{p-1})^T H^{-1} (\sigma_p - \sigma_{p-1}) d\Omega$$

Where σ_p and σ_{p-1} are the degree p and p-1 stress fields, H the Hooke matrix and Ω the volume of the structure. This technique gives an error measure, which has the following characteristics:

- a) The error is always underestimated.
- b) The error repartition is well estimated.
- c) If the solution is smooth, the effectivity index (ratio between the estimated and the exact errors) is very good (close to 1).
- d) If the solution is singular, the effectivity index seems to be independent of the mesh; it seems to depend only on the element degree.

So, if it is possible to determine the singularity level, it is also possible to correct the error estimation. The problem is to determine the singularity level of the finite element solution. Ideally, one likes to obtain this information from the comparison of the two hierarchical solutions. This can be done by taking benefit from the following observations [3] :

a) Singularities occur in high stress area and the singular point is characterized by high stress and stress gradient values.

b) The relative stress variation $\left(\frac{\sigma_p - \sigma_{p-1}}{\sigma_p} \right)$ is high in singular areas, and low where

the solution is smooth.

This new error indicator is able to predict both the errors of solution of degree p and p-1 and to deduce the global and local convergence rates of the solution. The error estimator corrected by an empiric law has been validated in many numerical tests and has always given satisfactory results. Using this technique, it has been possible to create an automatic hp-adaptive error control method, which is applicable for 2-D and 3-D problems.

Many applications in two and three dimensions will be presented.

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ADAPTIVE hp -FE MODELING FOR MAXWELL's EQUATIONS

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This is a progress report on the development of the hp Finite Element Method for steady-state Maxwell's equations in both interior and exterior domains. We shall report on the following theoretical and implementational issues.

- De Rham diagram for hp spaces.
- Interpolation error estimates for the hp discretizations, and the corresponding impact on stability and convergence analysis.
- Convergence analysis for a two-grid solver.
- Implementation of the Element Residual Method for 2D hp meshes.
- *3Dhp90_EM* - the three-dimensional hp -Adaptive Finite Element Package for Electromagnetics. Examples of 3D simulations (antenna and scattering problems).
- 3D implementation of the infinite element, numerical examples.
- Full wave analysis of nonhomogeneous waveguides. Existence and convergence results. Examples of simulations.
- A parallel implementation of the 2D two-grid solver on Cray 3TE.

Please visit my Web page: (<http://king.ticam.utexas.edu/People/Faculty/DEMKOWICZ/>) for more information and copies of our current reports.

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Recent Reports

- 1 L. Demkowicz, P. Monk, L. Vardapetyan, and W. Rachowicz. "De Rham Diagram for hp Finite Element Spaces" *TICAM Report 99-06*, submitted to *Mathematics and Computers with Applications*.
- 2 L. Demkowicz, P. Monk, Ch. Schwab, and L. Vardapetyan "Maxwell Eigenvalues and Discrete Compactness in Two Dimensions" *TICAM Report 99-12*, submitted to *Mathematics and Computers with Applications*.
- 3 L. Vardapetyan and L. Demkowicz "Full-Wave Analysis of Dielectric Waveguides at a Given Frequency", *TICAM Report 99-22*.
- 4 L. Demkowicz, A. Bajer, K. Gerdes, "3D hp-Adaptive Finite Element Package. Fortran 90 Implementation (3Dhp90)", *TICAM Report*, in preparation.
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ZOLTAN: A LIBRARY FOR DYNAMIC LOAD BALANCING

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In recent years, several high-quality software tools for static partitioning (e.g., Chaco [2], METIS [3], Jostle [6]) have been developed. By providing convenient and efficient partitioning, these tools have enabled the use of parallel computing for a variety of applications. In traditional finite element applications, for example, static partitioning tools are used as preprocessors to divide finite element meshes (and computational work load) among processors.

For adaptive finite element methods, however, static partitioning is not sufficient to provide good parallel performance. The per-processor work loads in adaptive methods can change as the mesh is locally refined (h -refinement) or the degree of the approximation is varied (p -refinement). For adaptive methods, dynamic load balancing is needed to redistribute work as a computation proceeds. General purpose tools for dynamic load balancing, however, have not been readily available. This fact is due to a number of differences between static and dynamic load balancing. Since static partitioners are often run as preprocessors, they can easily use file-based interfaces and data structures different from those of the applications. They may be implemented in serial and may use large amounts of memory and computing time. Dynamic load-balancing tools, however, must run along with the application. They must be implemented in parallel, and be fast and memory efficient to maintain the scalability of the application. They also need function-call interfaces and methods to extract data from the application's data structures. To be truly general-purpose tools, dynamic load balancers must maintain separation between the data structures of the load-balancing algorithms and the applications using them.

The Zoltan library [1] addresses many of the issues that arise in implementing dynamic load-balancing algorithms. Zoltan is a general-purpose dynamic load-balancing library containing a suite of load-balancing algorithms. To enable separation of data structures, Zoltan provides an object-oriented function interface between applications and load-balancing algorithms. Information needed by a load-balancing algorithm is obtained from the application through simple call-back functions. Several different load-balancing algorithms (including interfaces to ParMETIS [4] and Jostle [6]) are available in the library, and since a common data structure is not

required, new algorithms are easily added to the library. Both graph-based and geometric load-balancing algorithms are supported. We are also incorporating a heterogeneous computing model into Zoltan to allow partitioning across heterogeneous sets of processors or heterogeneous networks. The model represents a heterogeneous computer as a hierarchy, with the entire system's topology represented at the highest level, subcomponents' topology represented at intermediate levels, and individual processors' computing power and memory capacity represented at the lowest level. Our intent is to partition both the parallel computer and the application data in ways that minimize idle time on all processors and reduce communication across the slowest network connections.

In this talk, we will describe the interface and use of Zoltan. We will assess its performance and overhead costs in a 3D unstructured-mesh finite element code called MPSalsa [5].

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PARAMETERIZED MODELING OF MULTI-LAYERED STIFFENED SHELLS FOR p -VERSOIN ANALYSIS[†]

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Many structures of interest to aerospace and the defense sectors have shell like geometries that are stiffened by ribs and/or stringers. The stiffening components often vary in topology and geometry. For complex three-dimensional geometries, this poses a great challenge in the finite element modeling of such structures using continuum-based elastic shell elements using the p -version of the finite element approximation. The issues that need solution are two-fold:

1. geometric abstraction and modeling of the domain, followed by
2. satisfaction of displacement continuity along the stiffener junctions when p -hierarchic approximations are used.

The first issue is important because direct modeling of the junction details in three-dimensional space for individual stiffener configurations is impractical due to the complexity involved in geometric modeling and generation of valid curvilinear meshes. Independent specification of the in-plane and through-the-thickness interpolation degrees for the displacement field lead to non-conforming finite element spaces along the shell junctions.

This presentation will present modeling scheme where the shell and the stiffeners are geometrically idealized by model faces and edges, respectively. Each stiffener is abstracted by a set of geometric parameters that are sufficient to represent its three-dimensional geometry. This is coupled with an adaptation of the *mortar* scheme to enforce weak continuity of the displacement field along the stiffener junction. Three-dimensional examples are presented to show the use of the scheme in solving problems from elastodynamics and structural acoustics.

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**DEVELOPMENT OF p-VERSION HANDBOOK
SOLUTIONS FOR ANALYSIS OF COMPOSITE BONDED
JOINTS**

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The desire to reduce costs of composite aircraft structure has recently increased the importance of bonded joint analysis capabilities in the aeronautics industry. A new initiative has been formed, entitled the Composites Affordability Initiative (CAI), with the goal to decrease the recurring acquisition costs of composite airframe structure by 50%, thereby making composites more affordable for the next generation of fighter aircraft. Participants in the initiative include the Air Force, Navy, Lockheed Martin, Boeing, and Northrop Grumman. New validated analysis tools for composite bonded joints have been viewed as the key enabler for the application of innovative manufacturing and bonding technologies for airframe primary structure. Thus, improved analysis tools for design of bonded composite joints have become the focus of a CAI analysis tools team. This paper reviews the development history and describes new capabilities of a p-version Finite Element-based tool developed by a joint effort of the CAI team and a software provider.

The capability of performing rapid parametric sizing studies was one of the central goals of the team. It was desired to develop parameterized solutions for various bonded joint configurations, and validate failure predictions with CAI test data. These solutions would then be used by non-Finite Element experts to perform rapid, preliminary design sizing of composite bonded joints. Bonded joint analysis tools previously existed in the aeronautics industry (such as the industry standard A4EI code), but are limited to simple joint geometry (typically single, double, and step-lap joints) due to the one dimensional nature of the solution. Much more complex joints are being utilized to achieve affordability goals, thus solution techniques require at a minimum two-dimensional continuum elasticity analysis. Many finite element studies utilizing plane strain analysis appear in the literature, all utilizing h-element finite element models. Many utilized nonlinear models of the adhesive, and others included geometric nonlinearity. Different methods for handling the so-called "spew" fillet geometry, which is formed at the adhesive free edge, have also been considered. Uncertainty of the shape of this region has also been studied. Since singularities exist in the elasticity solution at free edges of composite layer interfaces, and also at reentrant corners, consideration of the convergence issues and error control in these regions is important. It is this author's opinion that more attention to error control in these regions is required, since failure tends to initiate in the vicinity of these singularities. Other important issues include the need for a modern, user-friendly interface, abilities for the user to build-in failure criteria,

advanced stress/strain extraction capabilities, and insensitivity to very large aspect ratios caused by the modeling of thin composite layers.

The CAI analysis tools team completed an extensive study of the state-of-the-art in the area of composite stress analysis tools, as would be applied to failure analysis of composite bonded joints. The software *StressCheck*, developed by Engineering Software Research and Development, Inc. (ESRD), was identified as a potential provider, and following an evaluation phase, was selected. This software offers a p- and hp-element formulation, which satisfies the need for error control, aspect ratio insensitivity, and also includes geometric and material nonlinearity. *StressCheck* already had an electronic handbook interface, which contained many of the required parameterization features. Once *StressCheck* became an approved CAI analysis tool for further development, CAI and ESRD worked closely to define *StressCheck* modifications required for the bonded joint applications. ESRD has implemented CAI requested software improvements through both industry contract funding and AFOSR sponsorship. This paper will review the modifications implemented into *StressCheck*, and discuss an example double-lap joint analysis. How the modeling philosophy for this joint differs from previous work will be discussed, including the handling of singularities, and error control. Actual failure criteria implemented, including stress/strain extraction logistics will not be reviewed, but examples of the general enhanced capabilities will be discussed.

This paper also illustrates an example of a very fruitful effort of the CAI team in identifying and filling an analysis tool requirement. An important conclusion is the need for industry users to work closely with software providers to precisely tailor software to the requirements of the problem and the focus user group. This experience highlights the key link being forged between aerospace industry researchers and software providers to develop next generation analysis tools.

ADAPTIVE DISCONTINUOUS GALERKIN METHOD WITH ORTHONORMAL BASIS

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We deal here with a high-order formulation of hyperbolic conservation laws using the discontinuous Galerkin method (DGM). Problems that we intend to solve are inherently transient and have discontinuities so that the DGM is an appealing alternative to stabilized finite elements or finite volumes.

We first propose a choice of an orthogonal basis for triangles and tetraedrons in order to diagonalize the mass matrix. This makes explicit time integration quick and easy. This basis is simply constructed by applying a modified Gram-Schmidt to an initial complete polynomial basis.

Monotone methods (i.e. methods that converges on a non oscillatory manner) for solving conservation laws are of first order of accuracy. One can use piecewise constant approximations of the conservative variables and achieve monotonicity. Hence, this method will give poor accuracy in smooth regions of the flow and, moreover, shocks will be heavily smeared due to large amount of numerical dissipation. We propose here a solution limiter that is applied after each time step so that oscillations are removed from the solution.

For the adaptivity, we propose an optimization based methodology for being able to adapt both element sizes (h) and polynomial degrees (p). We first defined what is an optimal $h - p$ distribution on the domain. The objective function is obviously the number N of degrees of freedom in the problem: we minimize N while keeping the error under a givel level.

Finally, we present numerical results for several computational examples in two and three dimensions using parallel tools available at RPI.

ARCHITECTURE-DEPENDENT FINITE ELEMENT COMPUTATION USING THE RENSSELAER PARTITION MODEL

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Adaptivity and parallel computation are essential to achieve solutions to computationally demanding three-dimensional problems. By automatically refining and coarsening finite element meshes (*h*-refinement), relocating meshes (*r*-refinement) and/or varying the method order (*p*-refinement) so that the effort is concentrated where resolution is needed, adaptive methods can increase time and space efficiency relative to traditional techniques. Parallel FEM computations are typically done by decomposing the spatial domain and distributing the underlying mesh and solution data across the system. With adaptivity, however, meshes and methods change, and the data structures must support dynamic mesh modification and migration. Architectural variations further complicate matters; thus, an efficient data distribution in a shared-memory environment may cease to be so in a distributed-memory, heterogeneous, or hierarchical environment. Partitioning must be done with some knowledge of the computational environment and we describe a hierarchical partition model that makes such information available.

The large range of problem sizes to be addressed and user computing environments available requires software that runs efficiently on computers ranging from single workstations to the largest parallel supercomputers. As an example, consider several common parallel computing environments. A program that is optimized for a distributed-memory environment with communication provided by a fast switch might partition the problem so that the workload is approximately equidistributed and the interfaces between the subdomains on different processors are small. A small load imbalance might be tolerated to reduce communication. Tolerating such a load imbalance may, however, not be worthwhile on an SMP workstation, where shared-memory communication is fast relative to a switch. An Ethernet-connected network of workstations (NOW) is at the opposite extreme: communication is very expensive. Thus, a larger imbalance may be accepted if some communication were avoided. With the same program executing in each environment, run-time information is needed to determine the optimal relationship between a balanced computational load and minimal communication. Complexity increases further with heterogeneous or hierarchical environments. A system containing ethernet-connected ATM clusters, each of which contains a number of SMP workstations, has three

levels of interconnection network. Two given processes may communicate by shared memory, ATM, or Ethernet. There is no need for great expense in partitioning data stored on the same SMP node. The ATM interface is relatively slower, but still fast enough to use an inexpensive partitioning and to maintain a close balance. Communication over Ethernet must be minimized, possibly by using a more expensive partitioning algorithm and by tolerating a larger load imbalance. Heterogeneous processors (different processor speeds or memory sizes) are also important.

The *Rensselaer Partition Model* (RPM) [4] allows a hierarchical partitioning of finite element data and provides run-time information about a parallel computing environment. RPM uses the mesh structures of the *SCOREC Mesh Database* (MDB) [1]; however, many ideas may apply to other systems. MDB provides an object-oriented, hierarchical representation of a finite element mesh with three-dimensional *regions* (elements) having links to their bounding *faces*, which are linked to their bounding *edges*, which are linked to their bounding *vertices*, making the structures suitable for both h and p -refinement [2, 3].

Each entity in a distributed mesh is uniquely assigned to a *partition*. The model is hierarchical, with each partition assigned to a specific *process* (the *process model*), and each process assigned to a *machine* (the *machine model*). Support for multiple partitions assigned to a process allows the use of partitions as a coarser structure for load balancing to (i) encourage locality of reference, (ii) enhance cache performance, and (iii) provide a means of performing out-of-core computation by swapping partitions in and out of memory. The process model represents the “live” state of a job and its computational and communication performance. The machine model maps processes to the actual processing nodes, and maps interprocess communication to the appropriate network or interface.

Information provided by RPM allows a partitioning procedure to avoid heavy communication at interprocess boundaries across slow interfaces. Many partitioning procedures minimize the communication volume of the total interprocessor boundary; however, this is not equivalent to minimizing the actual cost when communication speeds vary. Enhancements to existing partitioning algorithms, such as interprocess boundary smoothing and multilevel partitioning, and the development of new algorithms are possible with the availability of the information within RPM and are being explored.

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A POSTERIORI ERROR ESTIMATION FOR EVOLUTIONARY DISSIPATIVE EQUATIONS

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In recent years the numerical solution of dissipative partial differential equations has received an inflow of ideas from the theory of dynamical systems. An important contribution is the use of approximate inertial manifolds, AIMs, in the so-called nonlinear Galerkin methods [5]. One of the most often mentioned advantages of nonlinear Galerkin methods over classical Galerkin discretizations is their higher convergence rates.

In [3], an inexpensive novel technique, which we have called postprocessed method, to increase the accuracy and computational efficiency of spectral polynomial approximations was developed. The postprocess yields the same accuracy as a nonlinear Galerkin method, while the computational cost is kept nearly the same as standard Galerkin, so that the postprocessed method is more efficient than both standard and nonlinear Galerkin discretizations. The main idea in the postprocessed method is to calculate the AIM only once, when the time integration is completed, and add it to the Galerkin approximation.

In [4] we introduce a postprocess of the spectral element method [2] for time-dependent dissipative equations. Let Ω be a two dimensional domain with smooth boundary. We consider equations that can be written in the form:

$$(4) \quad \begin{aligned} \frac{\partial u(x, t)}{\partial t} &= \nu \Delta u(x, t) + R(u(x, t)) + f(x, t), \quad 0 \leq t \leq T, \quad x \in \Omega, \\ u(x, 0) &= u_0(x), \quad x \in \Omega, \quad u(x, t) = 0, \quad x \in \partial\Omega. \end{aligned}$$

where R can be a nonlinear convective term such as $R(u) = (u \cdot \nabla)u$, or a reaction type term such as $R(u) = g(u)$, for g a given smooth function.

The new approach may be summarize as follows: Suppose that the approximation is wanted at time $T > 0$. We first compute the Galerkin approximation based on a space of piecewise polynomials of degree N by numerically integrating the equation in the time interval $[0, T]$. Then, we postprocess by solving at the final time T a discrete linear elliptic problem on a space of piecewise polynomials of degree $M > N$. The rate of convergence of the resulting method is proved to be $\min(N^{-s-1}, M^{-s})$, so that, the method possesses a higher rate of convergence than the Galerkin method upon which the postprocess is applied. Since the elliptic problem in the higher order

polynomial space is solved only once, when the time integration is completed, the overcost of the postprocessed procedure is nearly negligible.

In this talk we will show that the postprocessed method can be used as an a posteriori error estimator for evolutionary dissipative equations [1]. More precisely, we will show that the error achieved using the spectral element method can be accurately estimated by calculating the L^2 or H^1 norm of the difference between the spectral element approximation and the postprocessed approximation that can be obtained from it. The postprocessed method, used as an a posteriori error estimator, reveals itself not only cheap and easily computable, but also able to give local and global information on the error of the numerical solution.

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HIGHER ORDER ACCURATE COMPLEX TIME STEP METHOD

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Recently, a new type of time step integration algorithms using complex time steps has been proposed. The dynamic responses are evaluated at specific sub-step locations and then linearly combined to yield solutions at the end of the time step. For linear problems, the algorithms are higher order accurate, unconditionally stable and have directly controllable numerical dissipation. It also maintains the accuracy of the particular solutions arising from the excitation. The complex time step method has been applied to structures under blast loading using a large time step and the algorithm can capture the high order loading accurately across the time step. The results are very accurate. However only linear problems were considered.

In this paper, the algorithms are extended to solve nonlinear problems. As an implicit method, the material state within the time interval has to be estimated to yield the solutions at the end of the time step. To maintain the dynamic equilibrium of the problem, nonlinear iterative solution techniques are necessary to control the accuracy of the solutions via a convergence check. The nonlinear iteration method chosen is the computationally efficient and commonly used pseudo force method. The pseudo forces are the internal forces accounting for the differences in the truly nonlinear system and the pseudo-linear system by the initial or fixed state. It is a linearization approach. These forces are updated at each iteration and treated as additional excitation on the structure.

When the iteration process is applied to the complex time step method, special treatments are required to handle the pseudo force in order to generate accurate numerical results. The linearization of the pseudo forces is removed by constructing the interpolation function to reflect the actual nonlinearity of the system. The pseudo forces are evaluated corresponding to the computed interpolation function. The nonlinearity is therefore restored within the time step. Since the complex time step method is a sub-stepping procedure, the pseudo force method is performed independently at the sub-step locations but the convergence check is carried out only after the sub-step responses are combined.

Several numerical examples are analyzed to demonstrate the applicability of complex time step method on nonlinear problems using the pseudo force method. It is observed that the complex time step method can be computationally much more efficient than the Newmark method and the Runge Kutta method when very accurate numerical solutions are required.

FULLY DISCRETE hp - FINITE ELEMENTS

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A fully discrete hp finite element method is presented. This method is based on the Spectral Galerkin Element Algorithm which we describe in detail. Our method combines the features of the standard hp finite element method (conforming Galerkin Formulation, variable order quadrature schemes, geometric meshes, static condensation) and of the spectral element method (special shape functions and spectral quadrature techniques). The speed-up (relative to standard hp elements) in the element computation results from our spectral algorithm that is based on sum factorization and making use of properties of the internal shape functions.

We present results on various problems from mechanics that show the computational advantages of this algorithm in terms of CPU time and convergence rates. We also obtain linear speed ups by performing the element computations in parallel, and we solve the sparse global linear system in parallel that results from static condensation of the element contributions.

JACOBI SPECTRAL METHOD

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Let $\Lambda = \{x|x| < 1\}$, $\alpha, \beta > -1$ and $\chi^{(\alpha, \beta)}(x) = (1-x)^\alpha(1+x)^\beta$. Denote by $(u, v)_{\chi^{(\alpha, \beta)}}$ and $\|v\|_{\chi^{(\alpha, \beta)}}$ the inner product and the norm of the space $L^2_{\chi^{(\alpha, \beta)}}(\Lambda)$. Similarly, we define the space $H_{\chi^{(\alpha, \beta)}}^m(\Lambda)$ with the norm $\|v\|_{m, \chi^{(\alpha, \beta)}}$. For any even integer $r = 2m$,

$$H_{\chi^{(\alpha, \beta)}, A}^r(\Lambda) = \{v|v \text{ is measurable and } \|v\|_{r, \chi^{(\alpha, \beta)}, A} < \infty\}$$

where

$$\|v\|_{r, \chi^{(\alpha, \beta)}, A} = \left(\sum_{k=0}^{m-1} \|\partial_x^{2m-k} v(1-x^2)^{m-k}\|_{\chi^{(\alpha, \beta)}}^2 + \|v\|_{m, \chi^{(\alpha, \beta)}}^2 \right)^{\frac{1}{2}}.$$

For any real $r \geq 0$, the space $H_{\chi^{(\alpha, \beta)}, A}^r(\Lambda)$ is defined by space interpolation. Next, for any non-negative integer μ ,

$$H_{\chi^{(\alpha, \beta)}, *, \mu}^r(\Lambda) = \{v|\partial_x^\mu v \in H_{\chi^{(\alpha, \beta)}, A}^{r-\mu}(\Lambda)\}, \quad H_{\chi^{(\alpha, \beta)}, **, \mu}^r(\Lambda) = \{v|v \in H_{\chi^{(\alpha, \beta)}, *, k}^r(\Lambda), 0 \leq k \leq \mu\}$$

with the following norms

$$\|v\|_{r, \chi^{(\alpha, \beta)}, *, \mu} = \|\partial_x^\mu v\|_{r-\mu, \chi^{(\alpha, \beta)}, A}, \quad \|v\|_{r, \chi^{(\alpha, \beta)}, **, \mu} = \left(\sum_{k=0}^{\mu} \|v\|_{r, \chi^{(\alpha, \beta)}, *, k}^2 \right)^{\frac{1}{2}}.$$

For any $\mu \geq 0$, we define the corresponding spaces by space interpolations.

Now let N be any positive integer, and \mathcal{P}_N be the set of all algebraic polynomials of degree at most N . Let $P_{N, \alpha, \beta}$ be the $L^2_{\chi^{(\alpha, \beta)}}$ -orthogonal projection from $L^2_{\chi^{(\alpha, \beta)}}(\Lambda)$ to \mathcal{P}_N .

Theorem 1. If $\alpha + r > 1$ or $\beta + r > 1$, then for any $v \in H_{\chi^{(\alpha, \beta)}, **, \mu}^r(\Lambda)$, $r \geq 1$ and $\mu \leq r$,

$$\|P_{N, \alpha, \beta} v - v\|_{\mu, \chi^{(\alpha, \beta)}} \leq c N^{\sigma(\mu, r)} \|v\|_{r, \chi^{(\alpha, \beta)}, **, \mu}$$

where $\sigma(\mu, r) = 2\mu - r$ for $\mu \geq 0$, and $\sigma(\mu, r) = \mu - r$ for $\mu < 0$.

If, in addition, $\alpha = \beta$, then $\sigma(\mu, r) = 2\mu - r - \frac{1}{2}$ for $\mu > 1$, $\sigma(\mu, r) = \frac{3}{2}\mu - r$ for $0 \leq \mu \leq 1$, and $\sigma(\mu, r) = \mu - r$ for $\mu < 0$.

If the coefficients of derivatives of different orders degenerate in different ways, then we can compare the approximate solutions with the exact solutions in certain Hilbert spaces. Let $\alpha, \beta, \gamma, \delta > -1$, $H_{\alpha, \beta, \gamma, \delta}^0(\Lambda) = L^2_{\chi^{(\gamma, \delta)}}(\Lambda)$, and

$$H_{\alpha, \beta, \gamma, \delta}^1(\Lambda) = \{v|v \text{ is measurable and } \|v\|_{1, \alpha, \beta, \gamma, \delta} < \infty\}$$

where

$$\|v\|_{1,\alpha,\beta,\gamma,\delta} = (\|v\|_{1,\chi^{(\alpha,\beta)}}^2 + \|v\|_{\chi^{(\gamma,\delta)}}^2)^{\frac{1}{2}}.$$

For $0 < \mu < 1$, we define the space $H_{\alpha,\beta,\gamma,\delta}^\mu(\Lambda)$ with the norm $\|v\|_{\mu,\alpha,\beta,\gamma,\delta}$ by space interpolation.

Let

$$a_{\alpha,\beta,\gamma,\delta}(u, v) = (\partial_x u, \partial_x v)_{\chi^{(\alpha,\beta)}} + (u, v)_{\chi^{(\gamma,\delta)}}, \quad \forall u, v \in H_{\alpha,\beta,\gamma,\delta}^1(\Lambda).$$

The orthogonal projection P_N^1 is such a mapping that for any $v \in H_{\alpha,\beta,\gamma,\delta}^1(\Lambda)$,

$$a_{\alpha,\beta,\gamma,\delta}(P_N^1 v - v, \phi) = 0, \quad \forall \phi \in \mathcal{P}_N.$$

Theorem 2. If $\alpha \leq \gamma + 2$ and $\beta \leq \delta + 2$, then for any $v \in H_{\chi^{(\alpha,\beta)},*,1}^r(\Lambda)$ with $r \geq 1$,

$$\|P_N^1 v - v\|_{1,\alpha,\beta,\gamma,\delta} \leq c N^{1-r} \|v\|_{r,\chi^{(\alpha,\beta)},*,1}.$$

If, in addition, $\alpha \leq \gamma + 1$ and $\beta \leq \delta + 1$, then for all $0 \leq \mu \leq 1$,

$$\|P_N^1 v - v\|_{\mu,\alpha,\beta,\gamma,\delta} \leq c N^{\mu-r} \|v\|_{r,\chi^{(\alpha,\beta)},*,1}.$$

If the functions vanish at one or two extreme points. we can define other orthogonal projections, see Guo [1, 2]

We now turn to some applications. We first consider the following problem

$$-\partial_x(k(x)\partial_x U(x)) + b(x)U(x) = f(x), \quad x \in \Lambda,$$

where $k(x) \geq 0, b(x) \geq 0$, and $k(x) \sim \chi^{(\alpha,\beta)}(x), b(x) \sim \chi^{(\gamma,\delta)}(x)$ as

$|x| \rightarrow 1$. Then we can use the Jacobi spectral method to solve this problem numerically.

Next, let $\tilde{\Lambda} = \{y \mid 0 < y < \infty\}$ and consider the logistic equation

$$\begin{cases} \partial_t V(y, t) - \partial_y^2 V(y, t) = V(y, t)(1 - V(y, t)), & y \in \tilde{\Lambda}, 0 < t \leq T, \\ V(0, t) = \lim_{y \rightarrow \infty} e^{-2y} \partial_y V(y, t) = 0, & 0 \leq t \leq T, \\ V(y, 0) = V_0(y), & y \in \tilde{\Lambda}. \end{cases}$$

Let $y(x) = -2 \ln(1-x) + 2 \ln 2$, $U(x, t) = V(y(x), t)$ and $U_0(x) = V_0(y(x))$. Then

$$\begin{cases} \partial_t U(x, t) - \frac{1}{4}(1-x)\partial_x((1-x)\partial_x U(x, t)) = U(x, t)(1 - U(x, t)), & x \in \Lambda, 0 < t \leq T, \\ U(-1, t) = \lim_{x \rightarrow 1} (1-x)^2 \partial_x U(x, t) = 0, & 0 \leq t \leq T, \\ U(x, 0) = U_0(x). \end{cases}$$

We can also use the Jacobi spectral method for this problem. We refer to Guo [2] for detail.

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MULTI-P PROCESSES AND PRE-CONDITIONERS

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A natural analogy to the multi-grid method which is based on the h-version of the finite element method is the multi-p method which is based on the p-version and hierarchic basis functions. Multi-p processes, including standard V-cycles, modified V-cycles, varying V-cycles, and fully multi-p methods, can be used as pre-conditioners. Methods for enhancing the performance of multi-p preconditioners by parallelization or replacing Gauss-Seidel smoothers with damped Jacobi smoothers are described. Results of numerical experiments are presented.

P-ADAPTIVE TECHNOLOGY: PERSPECTIVES ON ITS IMPACT AND FUTURE GROWTH

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The impact of p-adaptive technology reaches far beyond its contribution to automatic detection and control of numerical errors in finite element models. From a technical standpoint, p-adaptivity opens opportunities to address previously intractable challenges in diverse disciplines ranging from biomechanics to microelectromechanical systems (MEMs). On a broader perspective, experience with p-adaptivity has led to an increased role for CAE simulation in the mechanical design process. This improved understanding of design requirements will, in turn, underpin future progress at effectively deploying p-adaptive capabilities.

Over the past decade, p-adaptive technology has captured substantial mind share. In the late 1980s, Noetics emerged as the first significant commercial provider of p-adaptive finite element software. Noetics advertised that its software would allow engineers to focus less on generation of acceptable analysis results and more on their interpretation. Rasna Corporation then joined the field, identifying and executing a commercial strategy to enhance and promote the technology through its Mechanica product line. Rasna founders believed that the expertise required to control numerical error represented a key barrier to widespread use. By employing p-adaptivity to eliminate that barrier, they sought to accelerate the growth of the CAE simulation marketplace from a small community of expert technologists to a much broader community of designers and design engineers. Attracted by Rasna's rapid growth, Parametric Technology Corporation (PTC) acquired the company in 1995.

Today, p-adaptive Mechanica software represents approximately 12% of mechanical computer-aided engineering marketshare revenues – largely due to Mechanica's penetration of PTC's customer base. This growth has compelled other major suppliers of finite element analysis software to provide some level of p-adaptive support. All the major mechanical engineering analysis software suppliers – representing more than 60% of the market – now support some level of p-adaptive technology. In addition, a 1996 D.H. Brown Associates, Inc. (DHBA) survey suggests that 30% of all FEA practitioners have experience with p-adaptive capabilities. This population continues to expand.

Despite p-adaptive technology's proven effectiveness in numerical error control and considerable market exposure, its actual deployment in design practice has fallen short of expectations. More than just error control issues, achieving more extensive use faces major barriers such as the ability to translate design questions into well posed simulation activities. The expertise required to correctly interpret results further compounds this challenge.

As a result, the majority of designers still depend heavily on CAE simulation experts and design engineers, who spend at least half their time on engineering analysis. Role definitions and job performance metrics for designers also discourage more extensive use of computer-based simulation tools.

Consequently, more effective deployment of p-adaptive tools in practice will require software deployments that more precisely reflect the dynamics of design activities and capture industry-specific design requirements. The serious ramifications for design automation software based on these observations will also be discussed.

PRECONDITIONED ITERATIVE SOLVERS FOR COUPLED FEM-BEM SYSTEMS[†]

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We propose and analyze efficient preconditioners for solving linear systems arising from the hp -version with quasi-uniform meshes of the finite element/boundary element coupling.

For our first strategy we deal with a weak formulation that gives rise to non-symmetric stiffness matrices whose symmetric parts are positive definite. These systems are solved by the generalized minimum residual method. We study a preconditioner that amounts to a block-Jacobi method and a method that is partly given by a diagonal scaling. The first method requires $O(\log^2 p)$ iterations to solve the systems up to a given accuracy, whereas the second preconditioner, which is more easily implemented, requires $O(p \log^3 p)$ iterations. Here, p is the polynomial degree of the basis functions.

Our second strategy is to consider the symmetric formulation of the coupling of FEM and BEM. Here we have to deal with indefinite stiffness matrices and we use the minimum residual method as iterative solver. According to the structure of the Galerkin matrix we study 2- and 3-block preconditioners corresponding to Neumann and Dirichlet problems for the finite element discretization. In the case of exact inversion of the blocks we obtain bounded iteration numbers for the 2-block Jacobi solver and $O(h^{-3/4} p^{3/2})$ iteration numbers for the 3-block Jacobi solver. Here, h denotes the mesh size and p the polynomial degree. For the efficient 2-block method we analyze the influence of various preconditioners which are based on further decomposing the trial functions into nodal, edge, and interior functions. By further splitting the ansatz space with respect to basis functions associated with the edges we obtain a partially diagonal preconditioner. The penultimate method requires $O(\log^2 p)$ iterations whereas the latter method needs $O(p \log^2 p)$ iterations.

Numerical results are presented which support the theory.

[†]This work is based on a collaboration with Ernst P. Stephan and, in part, with Matthias Maischak (both at Universität Hannover, Germany).

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ON NONLINEAR AND HIGHER-ORDER PLATE ANALYSIS USING THE P-VERSION OF THE FEM

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We discuss an implementation of the p -version of the FEM for three-dimensional elastic problems under the restriction that the geometry of the structure is created by sweeping a two-dimensional domain orthogonally in the third direction z . Examples of such structures include most of all civil engineering structures, like plates, slabs, pillars, and beams. The z direction being perpendicular to the other two spatial directions, integrations in z direction can be performed independently and beforehand. Therefore, our model can also be viewed as a two-dimensional p -version finite element code for arbitrary higher-order hierarchical shell (plate) models. The user may even select different shell models *locally*. The selection of a specific shell model corresponds to a semi-discretization in thickness direction. In the in-plane discretization, the user may also select an arbitrary combination of h and p . We discuss the influence of the selection of the plate model on the solution, requirements for the in-plane discretization, generalization to multi-layer shell kinematics, curved shells, and practical applications.

In the second part of the talk, we discuss the application of the p -version of the FEM to elastoplastic problems. We start from plane problems and extend the discussion to the higher-order plate models. Discussion is limited to J_2 plasticity. Time permitting, the talk may be accompanied by practical demonstrations.

SPLINE VARIATIONAL THREE DIMENSIONAL STRESS ANALYSIS OF LAMINATED COMPOSITE PLATES WITH OPEN HOLES

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Independent polynomial spline approximation of displacement and interlaminar tractions is proposed for stress analysis in laminates with open holes. Spline approximation eliminates the inter element compatibility problems leading to unsatisfactory finite element results in the presence of field singularities. Spline approximation offers continuity of displacement, strain and stress fields within homogeneous domains preserving at the same time the advantages of local approximation, such as sparsity of the resulting system of equations. Three dimensional full field solution is obtained. Converged stresses and consistent boundary conditions, such as interlaminar traction continuity, are displayed. A close form asymptotic solution, valid in the vicinity of the hole edge at the interface of two orthotropic plies of arbitrary thickness has been developed to verify the spline approximation based full field solution. Excellent agreement has been observed for interlaminar stresses in a [45/-45]_s AS4/3501-6 laminate under uniaxial tension. The polynomial spline approximation, ideally suited for problems concerned with the singular solution behavior, has been applied to three dimensional stress analysis in practical composites containing tens of plies .

Related Publications

Larve, E. V., "Asymptotically Exact Stresses in Laminates With Rigid Fastener", *J. Composite Science and Technology*, in-press

Larve, E. V. and Pagano, N.J., "Singular Full-field Stresses in Composite Laminates with Open Holes", *Int. J. Solids Structures*, in press

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Larve, E. V., 1996, "Spline Variational Three Dimensional Stress Analysis of Laminated Composite Plates With Open Holes", *Int. J. Solids Structures*, 44, No. 14, pp. 2095-2118.

P-VERSION FEA IN THE AEROSPACE INDUSTRY REAL WORLD APPLICATIONS AND CHALLENGES

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There are many classes of problems in the aerospace industry for which p-version FEA is well suited. This paper highlights a broad range of problems that were solved using p-version FEA at two large aerospace companies. The method gives the aerospace industry a unique and productive tool to solve problems where high element aspect ratios are unavoidable, easy to use nonlinear analyses are desired, and multi-material and composite analyses involving strong stress singularities are encountered. Example problems will be shown and discussed in a general sense, and emphasis will be placed on the practical application issues faced by an analyst working in industry today.

Beyond the technical aspects of the method, this paper will also address the cultural issues that may be encountered while implementing p-version FEA. For example, the aerospace industry is well entrenched in traditional handbook methods and FEA codes. Breaking into this environment with revolutionary analysis tools takes a great deal of patience, persistence, and nurturing. Successful implementation strategies will be discussed.

While successful on many classes of problems, there are some problems in the aerospace industry that the p-version method cannot currently solve or handle with efficiency. This paper will discuss anticipated advancements and why they are important to the industry. Examples of needed features are: automated or semi-automated meshing algorithms for thin aerospace structures, robust methods to connect and analyze assembled parts, and the ability to combine and analyze unified part domains having dissimilar meshes.

Beyond the technical wish list items is the need to continuously educate the engineering community on FEA. This effort should note the differences in the underlying principles, capabilities, and reliability of the p- and h-version finite element methods. Much too often p-version FEA is misunderstood by analysts and underutilized by CAD software developers.

COMPUTATION OF ELECTROMAGNETIC SCATTERING WITH A NON-CONFORMING DISCONTINUOUS SPECTRAL ELEMENT METHOD

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Electromagnetic scattering problems are typically computed in the time-domain by approximating Maxwell's equations with low order finite difference or finite volume methods. These low order methods have large phase errors, which make them inefficient when applied to electrically large objects. High order finite difference methods, which have better phase properties, are not practical because they are difficult to apply in complex geometries or when material discontinuities are present [6].

Spectral element methods have features that make them attractive for scattering problems, when compared to finite difference and finite volume methods. They can be viewed as flexible extensions of the spectral collocation method [1], or as special cases of $h - p$ finite element methods. Like h -type finite element methods, complex geometries arising from scattering bodies or material interfaces that generate discontinuities can be treated easily with multiple feature-fitted elements. Within each element, the solution is approximated by an orthogonal polynomial expansion. Phase errors can be controlled either by decreasing the size of the elements or by increasing the order of the polynomials that approximate the solution [2].

In this paper we solve electromagnetic scattering problems by approximating the time-domain Maxwell's equations with a high-order discontinuous spectral element method (DSEM). The method is a collocation form of the discontinuous Galerkin method for hyperbolic systems. The solution is approximated by a tensor product Legendre expansion and inner products are replaced with Gauss-Legendre quadratures. For linear problems and straight-sided elements, the method is equivalent to the modal form. It is more convenient and simpler to code, however, when applied to non-linear problems or when the element sides are curved.

The fact that the solutions can be discontinuous at element faces makes a DSEM particularly suitable for the solution of hyperbolic systems in general, and of electromagnetic scattering problems in particular. The method does not require continuous metrics between elements, which simplifies the generation of the mesh. Because Gauss quadratures are used, special treatment of element corner points is not required. This simplifies programming significantly, increases efficiency, and permits the solution of problems with sharp corners without additional code. Finally, the

discontinuous method permits the accurate solution of problems with material discontinuities at any approximation order, as long as element faces are aligned with the material interfaces [3].

To increase flexibility of the DSEM, we introduce a mortar-element implementation [5],[4]. Mortars provide two benefits. First, they provide a means for coupling element faces along which the polynomial orders differ. This allows the flexibility to choose the approximation order within an element by considering only local resolution requirements. Secondly, mortars permit local subdivision of a mesh by connecting element faces that do not share a full side.

The final paper will present computations of some electromagnetic radar cross sections of bodies for which exact solutions are known. These include perfectly conducting bodies, dielectric bodies, and dielectric-coated perfectly conducting bodies.

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ON THE APPLICATION OF HIGH ORDER METHODS TO PROBLEMS IN STRUCTURAL DYNAMICS

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The finite element method is a widely used tool for the solution of problems in structural dynamics. Here, the most popular time integration algorithms are based on Newmark's method, which represents a family of second order finite difference formulas that can be applied to solve the semi discrete equations of motion.

A more recent development is the *Discontinuous Galerkin* method, allowing the unknown displacements and velocities to be discontinuous with respect to time. The method is unconditionally stable and of higher order accuracy. A two field formulation for elastodynamics problems using independent interpolations for the displacements and velocities was introduced by Hughes and Hulbert. The method was later advanced to general second order hyperbolic systems and extended to a one field formulation by the same authors. A two field formulation using a specialized efficient solver for the resulting linear equation system was developed by Li and Wiberg. Common to these methods is the use of low order, i.e. constant or linear, shape functions for the time discretization. Schötzau and Schwab extended the Discontinuous Galerkin method to use higher order shape functions in time. They provide the mathematical framework for the hp-version of the time integration algorithm in the context of first order, parabolic systems using a two field formulation.

In our present contribution, we apply the idea of p-extensions in time to the two field formulation for structural dynamics and wave propagation problems, i.e. to second order, hyperbolic systems. After introducing the basic notation, the high order time Discontinuous Galerkin method with hierarchical shape functions in time is outlined. Following ideas of Schötzau and Schwab, we develop an efficient algorithm to decouple the global time equation system allowing for a stepwise subsequent solution. In numerical examples the efficiency of the method is compared to results from Newmark's method and limitations of the current two field formulation are discussed. Finally we show the coupling of the generically implemented time integration algorithm to a commercial finite element package.

**THE APPLICATION OF p-VERSION FINITE ELEMENT
SOLUTIONS FOR ADVANCED FATIGUE ANALYSIS**

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The Finite Element Method (FEM) is one of the primary tools in advanced fatigue analysis for assessing the stresses of complex machined parts under cyclic loading. Since the fatigue life is very sensitive to stress concentration, high accuracy of FEM stress solutions is always required. The utmost importance of any fatigue stress analysis using FEM is to accurately locate the maximum stressed area and determine the notch stresses. The part may be subjected to complex multi-component loading time histories and in many cases, the notch stresses are multi-axial. For this purpose the p-version FEM has a great advantage over the h-version FEM. This paper summarizes the applications of the p-version FEM software (MECHANICA, STRESSCHECK) in the notch stress computations for crack initiation life prediction of aircraft structures. The model solutions were validated using a procedure that includes measurements and judgement. Ten practical cases are presented covering various parts of the aircraft. Some comparisons of the p-version FEM solutions versus solutions from other methods are also presented. Fatigue life computation accuracy due to variation with model results is presented. Recommendations are presented for future FEM development

MULTIPLICATIVE SCHWARZ ALGORITHMS FOR THE
GALERKIN BOUNDARY ELEMENT METHOD OF THE
FIRST KIND^s

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We study multiplicative Schwarz methods for the hp -version with quasi-uniform meshes of the boundary element method applied to hypersingular and weakly singular integral equations on open curves and open surfaces.

Proving a strengthenend Cauchy Schwarz inequality for our 2-level subspace decomposition and utilizing results for the corresponding additive Schwarz method we show that the rate of convergence for our 2-level multiplicative Schwarz method grows only like $1 - C \log^{-2} p$ towards one as p increases. As a consequence, the number of iterations necessary to achieve a given accuracy of the discrete solution grows only logarithmically in p . Here, p is the polynomial degree of the basis functions.

Computational results are presented for the hp -version of the Laplacian and the Helmholtz operator which support this theory.

^sThis work is based partly on a collaboration with Ernst P. Stephan (Universität Hannover, Germany) and Thanh Tran (Australian National University, Canberra, Australia).

NON-CONVENTIONAL DUAL ANALYSES IN THE CONTEXT OF ERROR ESTIMATION

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Introduction

One of the principal methods that has been developed for error estimation in elastostatics relies on knowledge of both a conforming or kinematically admissible solution, and an equilibrating or statically admissible solution [2,4]. These lead to bounds to global errors, and if they are close to each other then very effective error estimates are possible. Whilst dual solutions may be derived from separate analyses of complete models, it is also possible to derive one solution from the other using localised analyses with the aim of obtaining effective error estimates at lower computational efforts. Dual solutions derived in this way have been termed non-conventional, and this paper is principally concerned with the methods of deriving such solutions using hybrid elements which are also non-conventional [3]. An alternative approach is also proposed based on a Trefftz patch recovery method for locally smoothing displacement or stress fields, and for providing estimates of local errors [10]. The patches may consist of conforming or equilibrating elements such as the hybrid type.

Elements are considered in the context of modelling 2D, 3D, or plate problems where further simplifications are made in modelling through thickness behaviour. The primary model is considered with a variety of possible types of elements all of which have a p -type formulation [1], where p refers to the degree of displacement or stress polynomials.

Conforming displacement elements which include nodal variables at vertices.

Equilibrium solutions are derived by resolution of nodal forces followed by recovery of consistent tractions [6]. These tractions are then used as Neumann type boundary conditions for local element analyses - using hybrid equilibrium elements in place of the original elements. The force resolution stage does not provide unique tractions, various options are possible and the use of Maxwell force diagrams is presented as an engineering solution for the 2D membrane and plate bending cases. The 3D case is presented in graph-theoretic terms [5] to represent the more complex local connectivities. Since the solutions are not unique, there is scope for optimisation schemes and an hierarchical form of flexibility method.

Conforming hybrid elements with side variables

These elements may be viewed as a reformulation of the conventional conforming element, but with side traction modes and dual displacements replacing nodal variables. This implies that equilibrating tractions are more directly derived, but the presence of spurious static modes [9] leads to locking and tractions which are indeterminate both from the static and kinematic points of view. This again implies that tractions are not necessarily unique, and so there is scope for local optimisation which is discussed in the paper.

Equilibrium hybrid elements with side variables.

In this case the primary model produces an equilibrating solution for stress [7]. A dual solution involves recovery of compatible displacement fields. A method has been proposed [8] based on element by element fitting of displacement fields and positions, followed by smoothing of displacements at interfaces. An alternative is proposed in this paper based on retaining the interface displacements derived from the hybrid equilibrium model and replacing the original elements with higher degree hybrid displacement elements. In this way the ends of the interfaces are made to conform with a compatible displacement field. Again solutions are not unique for the displacements of the vertices, and there is scope for local optimisation of these displacements. The question of spurious kinematic modes is also addressed.

Trefftz patch recovery scheme for stresses or displacements.

A patch recovery scheme based on Trefftz solutions for smoothing of discontinuous displacement and/or stress fields has been proposed [10]. This is discussed in the context of the primary models considered above with numerical examples for Reissner-Mindlin plates with Trefftz fields up to degree 4. The Trefftz formulations for a patch are presented as for a "hermaphroditic" patch element to emphasize the dual features of Trefftz elements. Local error estimation in a patch using the Trefftz solution as a reference solution is also considered

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HIERARCHICAL MULTILEVEL GRADED hp-FE ALGORITHMS WITH PARALLEL PROCESSING BASED ON HEXAEDER ELEMENTS WITH NODE REGULAR REFINEMENTS

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For the mathematically ensured, cost-effective, flexible and automatic computation of structural mechanical problems with error bounds and tolerances, adaptive finite element meshes (h -adaptivity) and elements with different Ansatz-order (p -adaptivity) as well as dimension (d -adaptivity) are desirable. Isotropic h -refinement and anisotropic p -refinement (according to adequate anisotropic error estimators) is realized for the hierarchical Legendre tensor product ansatz functions. Anisotropy is an important issue for efficient 3D-FE-solutions in general. Because of the high numerical effort, the use of parallel computers is adequate. This lecture presents object oriented data structures and algorithms which support these adaptivities.

In most problems of structural mechanics we have to analyze thin-walled and at least partially orthogonal geometries. These can be discretized much better with hexaeder elements than with tetraeder elements. We describe a refinement algorithm which adapts hexaeder meshes in a node regular way, i.e. without hanging nodes, where degenerated hexaeders (\rightarrow trieders) appear temporarily. They disappear at further refinement which - together with adequate numerical integration - ensures the full numerical accuracy.

Three topological distinct elements are needed to achieve the locking-free transition from two dimensional to three dimensional areas of discretizations. Together with these hybrid of shell and 3D continuum elements, they permit the combination of both models in order to get the complete three dimensional solution in disturbed layers without the necessity of using 3D-continuum elements for regular thin-walled subdomains. The transition elements have continuum nodes with three degrees of freedom as well as shell nodes with five degrees of freedom and couple the continuum with the shell model in a purely kinematical and smooth way.

Furthermore we show a new discretization concept based on an hierarchically graded multilevel ansatz. Within this ansatz individual p -orders of the elements can be used on each h -refinement level. This is done in a decreasing manner from the coarse to the finest mesh level in order to balance the approximation quality of the total ansatz and the computational effort for solving the resulting system of equations.

To solve the system of equations in a parallel efficient way, iterative multilevel preconditioned Krylov space methods like the cg and Lanczos method in the symmetric and the BiCgStab and GmRes method in the non-symmetric case are used. With the presented hierarchical Legendre Ansatz the condition number compared with the Lagrangian interpolation polynomials is noticeably smaller, and the transfer operator needed, e.g. by the BPX preconditioning, becomes trivial which improves the efficiency significantly.

In order to keep a general and simple program structure for implementing these complex algorithms (h-, p-, and d-adaptivity, distributed data) they are coded in the object-oriented language C++ within our parallel FE-program ParaFep. This program also performs the dynamical load balancing which is needed within adaptive parallel computations.

Some large dimensioned numerical examples illustrate the robustness and efficiency of the presented methods.

HIGHLY ACCURATE MODE-SEPARATED ENERGY RELEASE RATES FOR DELAMINATED COMPOSITE LAMINATES

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Following Irwin's argument, the energy release rates are computed by using nodal forces ahead of the crack front and relative displacements behind the crack front. Furthermore, the quarter-point singular elements were used for most finite element computations. In this paper, the energy release rates are computed by the methods that are different from those traditional methods.

Recently, Babuska and Oh introduced a new approach, called the method of auxiliary mapping (MAM), that can effectively handle singularities in the framework of the p -Version of the FEM. In this paper, this new approach is modified so that it can yield highly accurate mode-separated energy release rate for the delaminated composite laminates. The results obtained by this method are about 10% more accurate than most known results. It is known that because of oscillatory behavior of displacements near the crack tip, G_I and G_{II} for delaminated composite laminates do not converge. However, it will be claimed that in practical point of view, G_I and G_{II} of interfacial crack between orthotropic composite materials are virtually convergent. In this paper, total energy release rate will directly computed by differentiating the total strain energy with respect to the crack length.

EQUILIBRATED RESIDUAL SPATIAL h- AND p- ERROR ESTIMATORS FOR ELLIPTIC, PARABOLIC AND ELASTOPLASTIC PROBLEMS

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First, the equilibrated error estimator with guaranteed upper bound property for the error in the energy norm will be given. The error estimator is gained by solving local Neumann problems with previously calculated equilibrated boundary tractions regarding to the current solution. The equilibrated tractions t^* have to fulfill on the one hand the consistency condition, i.e. the global equilibrated residual $\mathcal{R}^*(v)$ has to be equal to the given global residual $\mathcal{R}^*(v) = \mathcal{R}(v) \forall v \in V \subset H^1$. This leads to the C^0 -continuity condition of the equilibrated boundary tractions in normal direction at the finite element boundaries. On the other hand the solvability of the local Neumann problem leads to the equilibrium condition for the equilibrated residuals $\mathcal{R}_{\Omega_e}^*(v) = 0 \forall v \in Z(\Omega_e)$. Herein is $Z(\Omega_e)$ the kernel (rigid body displacement modes) of the local element bilinear form. Examples will explain the efficiency of the error estimator, see e.g. [1, 2, 3].

Second, for parabolic problems a special norm in space and time will be presented in the frame of semi-discrete finite element method, discrete in space and sufficiently exact (quasi continuous) in time. Referring to this norm a guaranteed upper error estimator will be presented which used the results from the previous explained error estimator for elliptic problems, see [4]. This error estimator is computed again by solving local Neumann problems with integration over the time to take into consideration the accumulation of the error in time direction. Two examples will be given for a 1D- and a 2D- parabolic problems in application to elasticity.

Third, for problems with elasto-plastic material behavior a guaranteed spatial error estimator will be presented under the assumption that the current tangent is sufficiently exact, see [5]. An example will present the efficiency of the error estimator.

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SOLUTION OF THE CONTACT OPTIMIZATION PROBLEMS OF CYLINDRICAL BODIES USING HP- FEM*

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This lecture deals with mechanical contact problems using displacement based p -versional finite element method. In the contact region both the normal stress and the tangential stresses may have singularities as well as jumps in their derivatives. The boundary of the contact zone is unknown *a priori*. Therefore the locations of the singularities in the normal and tangential stresses and jumps of their derivatives are also unknown.

Concerning the finite element discretization we have problem of category C [1] that is the mesh in 2D cannot be constructed so that the points where the solution is not analytic are not in vertices. When the p -version is used then the accuracy is typically high enough for the singularities to induce oscillations in the numerical solutions. For the treatment of axially symmetric contact problems an adaptive finite element method has been developed by which the category C problem is converted into category B.

The mesh is adjusted in the iteration process such that the boundary of the contact zone and the boundary of the stick-slip zones are nodal points, allowing the jumps in the derivatives to be represented in the discretized problem [2]. The accuracy can be increased by the enrichment of the finite element space with special singular functions [3].

In the present work it will be assumed that the displacements and deformations are small, the material of the contacting bodies are elastic. The augmented Lagrangian technique is used for the solution of the contact problem.

The p -version of the finite element method allows relatively coarse mesh. The positioning of the nodal points in the border of the contact zone is performed in one or two phases depending on the predefined accuracy to be achieved. The first phase is rough positioning of the contact points its aim is to achieve the situation when each of the integration points of the contacting elements is in contact. In the second phase a fine adjustment of the location of nodal points is performed on the bases of monitoring different indicators (the potential energy, the smoothness of the contact pressure/ normal displacement and contact pressure at the border point of the contact region).

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In case of frictional contact problem a nodal point in the contact region is positioned to the border point of the stick and slip zones in order to satisfy the Coulomb friction law.

The efficiency of the proposed method will be demonstrated through couple of examples. There will be examples in which the right/left side or both sides of the contact zone will be positioned.

Two types of mechanical contact optimization problems will be investigated:

1. optimization by the controlling of the contact pressure distribution,
2. maximization of the global mechanical parameter concerning a machine tool operation.

The contact pressure generally may have singularity in the contact zone without shape optimization. Our aim is to control the contact pressure to have smooth contact pressure as defined by a control function.

The control can be described by the following formula

$$\chi(s) = v(s)p_{\max} - p(s) = 0, \quad s \in \Omega_c$$

where s is the curve coordinate along Ω_c control zone, $v(s)$ - controlling function prescribed by the user, p_{\max} - maximum of the contact pressure, $p(s)$ is the contact pressure. On the complement subregion of Ω_c that is on the no control part $\Omega_{nc} = \Omega_{opt} \setminus \Omega_c$ we have an inequality $\chi(x) \geq 0$. We have prescribed the controlling function to be C^1 class function, actually constant in the middle of the contact zone and Hermite polynomials at the border zones.

The optimization problem is a restricted linear programming problem. For the solution a special iterational algorithm has been developed which proved to be faster than standard mathematical program package.

The following problems will be analyzed:

1. Contact optimization problems of cylindrical bodies in overlapping case.
Computation of the stress state due to assembly process, considering the heat effects.
2. Contact shape optimization of cylindrical clamping bodies when maximization of the torsion moment transmission is to be achieved.

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ON THE DEVELOPMENT OF LARGE SCALE
PARALLEL ADAPTIVE 3D CODES: DATA
MANAGEMENT SCHEMES AND MULTI-LEVEL
ITERATIVE SUBSTRUCTURING SOLVERS

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Successful large scale (capable of solving problems with millions of unknowns) parallel adaptive hp codes require efficient data management schemes, good adaptive strategies and fast/robust solvers that can reliably solve irregularly sparse and often poorly conditioned linear systems. We begin by describing recent work on developing a code infrastructure that performs the necessary data management, partitioning, parallel refinement/enrichment and dynamic load balancing. Central to this development is the use of a self-organizing data management scheme based on balanced tree data structures on each processor composed into a distributed list structure. Integrated with this structure we have also developed a class of solvers described next.

In past work, on these methods we have developed iterative substructuring based solvers and coarse grid hierarchical preconditioners that are quite efficient for 2D elliptic problems. We note that our use of automatic partitioning algorithms and unstructured grids precludes the nested coarse grids that have been traditionally used. Our coarse grid functions are derived naturally as linear combinations of fine grid functions. For such problems, we have proved that they have condition numbers that grow at rates that are polylogarithmic in the mesh discretization parameters. These solvers were also naturally extended to the Stokes equations. However, on more complex 3D problems, such efficiencies are not always obtained. In related work we have also developed a multi-level substructuring type solvers based on a good fill-reducing ordering using a space-filling curve passing through the element centroids. This ordering also constitutes the basis for our data indexing and data management schemes. This solver is very robust, but not as efficient as the iterative substructuring solvers.

We report here the development of hybrid solvers that combine the advantages of both by automatically switching from one algorithm to the other, depending on the problem, to produce robust solvers that work reliably on a large class of 3D problems. The solver starts with a substructuring on the lowest levels (bubble and edge function degrees of freedom) that are local to a processor, and switches to a preconditioned iterative solver at a level of the hierarchy that requires access

to non-local data. The preconditioner is a very coarse grid and diagonal blocks corresponding to the remaining degrees of freedom. If the solver exhibits difficulty in converging, we expand the definition of the preconditioner to include more and more degrees of freedom until we reduce to the multi-level substructuring solver described earlier. Further, since the choice of which solver is more efficient on a multi-processor architecture is dependent on the relative costs of computation and communication it is possible to tune these solvers to particular machine architectures by selecting the number of levels of substructuring to use.

THE APPLICATION OF P-VERSION FINITE ELEMENT METHODS TO FRACTURE-DOMINATED PROBLEMS ENCOUNTERED IN ENGINEERING PRACTICE

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This paper details APES's experience applying the higher order (*p*-version) finite element software StressCheck (ESRD, Inc., St. Louis, MO, USA) to highly specialized and difficult problems in structural life predictions, namely in predictions of aircraft structural fatigue life using crack growth analyses. A typical crack growth simulation uses the geometry and loads in a structure to compute Mode I Stress Intensity Factors (SIFs) K_I . The SIFs are used to derive crack growth rates, da/dN , which can be interpreted as an intrinsic material behavior, in addition to properties such as Young's modulus and Poisson's ratio, for a broad range of environments. The crack growth rate is used to calculate an increment in the crack length (da) for a given number of fatigue cycles (dN). The increased crack length is computed, the new SIF estimated, and the simulation continues until fracture of the structure is predicted. Typically, a crack growth simulation will require thousands of cycles (dN). Fortunately, accurate and reliable stresses and stress intensity factors derived from StressCheck simulations, obtained quickly and efficiently, enable the basic procedure to be successfully applied to development of advanced life prediction methods.

Many FEA software packages could have been used to obtain engineering data such as SIFs and stresses. However, the advanced numerical methods incorporated into higher-order finite element methods such as StressCheck allow the analyst to obtain more accurate and reliable results with convenient error checking, numerical convergence of engineering data such as maximum stress and strain and SIFs, stress contours, and deformations. In addition, greatly shortened analysis schedules are realized through the use of procedures such as the automatic extraction of SIF parameters by using the Contour Integral Method (CIM). The versatility of higher-order methods is demonstrated by some examples that we have encountered in engineering practice—multi-site damage scenarios and severely corroded plates.

Corrosion is a difficult structural problem to solve, requiring innovative solutions in order to overcome substantial challenges. Often the engineer will have a myriad of structural effects to understand, such as stress risers caused by corrosion topographies, bending stresses induced by corrosion by-product build-up between adjacent parts, and multiple cracks in the structure induced by corrosion and fatigue cycling. The advances in *p*-version FEM have provided the necessary inputs to crack growth analyses used to predict the effects of corrosion on aircraft structural life. Applications of advancements made in StressCheck have enabled incorporation of the key analysis characteristics necessary to model the effects of corrosion on structural life predictions.

ADAPTIVE DISTRIBUTION OF DYNAMIC GRID HIERARCHIES

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This paper presents the design and evaluation of an adaptive, system sensitive distribution/load-balancing framework for distributed adaptive grid hierarchies that underlie parallel adaptive mesh-refinement (AMR) techniques for the solution of partial-differential equations. The framework uses application and system state information to select the appropriate distribution scheme at run-time. The selection is driven by an application-centric performance characterization of dynamic partitioning and load-balancing techniques, and is governed by rules defined in a policy database. The primary motivation for the framework is the design and development of policy driven tools for automated configuration and run-time management of distributed adaptive applications on dynamic and heterogeneous networked computing environments.

Dynamically adaptive methods for the solution of partial differential equations that employ locally optimal approximations can yield highly advantageous ratios for cost/accuracy when compared to methods based upon static uniform approximations. These techniques seek to improve the accuracy of the solution by dynamically refining the computational grid in regions of high local solution error. Distributed implementations of these adaptive methods offer the potential for the accurate solution of realistic models of important physical systems. These implementations however, lead to interesting challenges in dynamic resource allocation, data-distribution and load balancing, communications and coordination, and resource management. The overall efficiency of the algorithms is limited by the ability to partition the underlying data-structures at run-time so as to expose all inherent parallelism, minimize communication/synchronization overheads, and balance load. A critical requirement while partitioning adaptive grid hierarchies is the maintenance of logical locality, both across different levels of the hierarchy under expansion and contraction of the adaptive grid structure, and within partitions of grids at all levels when they are decomposed and mapped across processors. The former enables efficient computational access to the grids while the latter minimizes the total communication and synchronization overheads. Furthermore application adaptivity results in application grids being created, moved and deleted on-the-fly, making it necessary to efficiently re-partition the hierarchy so that it continues to meet these goals.

Moving these applications to dynamic and heterogeneous networked computing environments introduces a new level of complexity. These environments require selecting and configuring application components based on available resources. However, the complexity and heterogeneity of the environment make selection of a "best" match between system resources, application algorithms, problem decompositions, mappings and load distributions, communication mechanisms, etc., non-trivial. System dynamics coupled with application adaptivity makes application configuration and run-time management a significant challenge. Clearly there is a need for "smart" tools that can automate the configuration and management process.

This paper first presents an application-centric characterization of distribution mechanism for AMR applications on heterogeneous (and dynamic) cluster computing environment. It then describes the design, implementation and evaluation of an automated application configuration and management framework that dynamically adapts distribution and load-balancing mechanisms based on current application and system state.

THE P-VERSION FOR LINEAR AND NON-LINEAR ANALYSIS IN THREE DIMENSIONS

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It is today widely accepted, that the p- and hp-version of the finite element method is superior to the classical h-version with respect to accuracy and computational efficiency, if linear, elliptic problems in two dimensions are considered. Yet, many researchers still doubt, that these advantages can also be observed for more complex non-linear or three-dimensional problems. It will be show in this contribution, that the superiority of the p-version may be even more dramatic in these cases.

We will first focus on the aspect of geometric modelling in three dimensions. The well-known blending function technique for mapping the geometry of p-elements offers the possibility to completely separate all geometric computation involved in a finite element analysis from the non-geometric parts. This separation allows to design a *distributed software system*, where the geometric model of a CAD-program is directly linked in a *client-server software structure* to a finite element kernel. This concept offers the advantage of using all state-of-the-art CAD-techniques like geometric editing or parametric design in a finite element analysis. The increase of efficiency for practical work can be dramatic, as such a system for computer integrated engineering relieves a user from the burden to transfer geometric data from CAD to FEA, which is usually very time-consuming, even if only some geometric parameters of the model change.

The paper will describe the software structure of our integrated system in detail. Some basic concepts of element matrix computation will be reviewed in the light of the blending function technique. We will outline the advantages of this approach for design processes in civil engineering applications. Special emphasis will be laid on spatial constructions like shells and combinations of shell and three-dimensional solid models.

The second part of the paper will address physically non-linear problems. As a model problem we will consider the rate independent plasticity with non-linear isotropic hardening, also known as the flow theory of Prandtl-Reuss. Small strains will be assumed and a *von Mises* yield criterion will be applied. As an integration algorithm for the rate-independent equations of plasticity we will consider an implicit Euler scheme by applying a return-mapping algorithm.

Finally, we will compare our p-version approach to the classical h-version for some benchmark problems as well as for more complex real life problems. Applications for civil engineering constructions as well as for bio-mechanical problems will be presented.

CURRENT USAGE OF TWO P-VERSION FINITE ELEMENT ANALYSIS CODES AND PROPOSED CONCEPTS FOR FUTURE GROWTH AND CAPABILITY

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The purpose of this paper is to present the current usage of Finite Element Analysis (FEA) tools at Boeing - St. Louis as well as recommended enhancements for increased capability. Two types of p-version FEA codes will be discussed; StressCheck by Engineering Software Research & Development and Pro/MECANICA by Parametric Technology Corporation.

The paper will focus on conventional strength and fracture mechanics applications that are performed by structural engineers. Current applications include design trade studies, manufacturing support, and advanced design efforts. Included is a comparative discussion of strengths and weaknesses of the various features available in each code.

The paper will also propose concepts to promote future growth and increased capability. Discussion is provided on recommended feature modifications and topics for new analytical tool capability. The purpose of this proposal is to initiate dialogue within the FEA software community to encourage development growth. A path will be suggested for both software designer and customer to remain ahead of their respective competition by developing new ways to increase capability, improve accuracy, and reduce analysis time.

A POSTERIORI ERROR ESTIMATES FOR A
DISCONTINUOUS GALERKIN METHOD APPLIED TO
ELLIPTIC PROBLEMS

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A posteriori error estimates for locally mass conservative methods for subsurface flow are presented. These methods are based on discontinuous approximation spaces and referred as Discontinuous Galerkin methods. One can show an *a priori* exponential rate of convergence for elliptic problems. Here, the hp adaptivity is investigated for flow problems in $2D$. Two types of global error estimators are derived: an explicit L^2 error estimator that can be used as an error indicator and an implicit H_0^1 error estimator that is based on the norm of residual.

FEM - ERROR ESTIMATES FOR STATIONARY NAVIER-STOKES EQUATIONS IN NONSMOOTH DOMAINS

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The first part of the lecture is concerned with a short review on regularity results for solutions of mixed boundary value problems for systems of stationary quasilinear and semilinear partial differential equations in domains with conical points, edges and vertices [1]. We introduce weighted Sobolev spaces with attached asymptotics generated by the asymptotical expansions of solutions of corresponding linearized problems near the boundary singularities. We formulate conditions which guarantee that the solutions of the nonlinear problem have the same asymptotical structure as the solutions of the linearized problem.

In a second part we apply these abstract investigations to general boundary value problems for the Navier-Stokes equation which model complicated flow conditions on the boundary. The resulting regularity results are crucial for an error analysis of finite element methods. Error estimates for uniform and graded meshes are derived [2].

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P-VERSION INTERFACE ELEMENTS IN GLOBAL/LOCAL ANALYSIS

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When performing global/local analysis, the issue of connecting dissimilar meshes often arises, especially when refinement is performed. One method of connecting these dissimilar meshes is to use interface elements. Curve interface elements, to connect dissimilar p-element edges along a curve; surface interface elements, to connect dissimilar p-element faces over a surface; and shell-to-solid transition interface elements, to connect dissimilar p-element edges with p-element faces, have been implemented in MSC/NASTRAN. The background, theory, and implementation are being presented herein, along with examples. These three interface elements comprise a complete set of interface tools for global/local analysis.

1. Introduction

The problem of connecting dissimilar meshes at a common interface is a major one in finite element analysis. One method of connecting these dissimilar meshes is to use interface elements.

1.1 Applications

The interface technology can be divided into two distinct types. The first, boundary technology, connects large numbers of elements along a single geometric curve or surface, as specified by the user. Primary applications facilitate: global/local analysis, where a patch of elements may be removed from the global model and replaced by a denser patch for a local detail, without having to transition to the surrounding area; and component assembly, where meshes built by different engineering organizations may be connected, such as a wing to the fuselage of an airplane. The second, interior technology, connects a few elements together in a mesh transition region, without user specification. Primary applications enable: automeshers, which may be required to transition between large and small elements between mesh regions; and h-refinement, where subdivided elements may be adjacent to undivided elements without a transition area.

1.2 Previous Methods

Much work has been done to resolve the element interface problem, with most of the efforts concentrating on moving the nodes or writing multi-point constraint (MPC) equations on the interfaces. However, both of these methods have difficulties that prevent them from being practical for the general problem.

1.3 Current Method

The need and applications for reliable interface technology are great. NASA Langley Research Center has developed a method for analyzing structures composed of two or more independently modeled substructures, based on a hybrid variational formulation with Lagrange multipliers, and applied it to global/local demonstration problems for one-dimensional and two-dimensional interfaces. NASA has also developed the technology for a solid-to-shell transition element for use with composites, and has combined it with the one-dimensional interface element.

Under terms of a cooperative agreement between MSC and NASA, MSC has extended and implemented this interface technology into MSC/NASTRAN for p-element edges along a geometric curve, for p-element faces over a geometric surface, and for solid p-element faces and shell p-element edges. This agreement is part of NASA's continuing effort to transfer technology into the mainstream of industry as an aid in developing competitiveness in the worldwide market.

PARALLEL AND ADAPTIVE MULTI-CONSTRAINT GRAPH PARTITIONING¹

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Algorithms that find good partitionings of unstructured and irregular graphs are critical for the efficient execution of a wide range of scientific simulations on high performance parallel computers. However, for many types of computations that are now commonplace, such as adaptive mesh, multi-phase, and multi-physics simulations, the traditional graph partitioning formulation is not adequate. For these, more generalized graph partitioning formulations and algorithms are needed. In this talk, we describe some important classes of scientific simulations that require new formulations of the graph partitioning problem, we discuss these requirements, and we describe new, generalized formulations of the graph partitioning problem as well as algorithms for solving these problems.

For large-scale scientific simulations, the computational requirements of techniques relying on globally refined meshes become very high, especially as the complexity and size of the problems increase. By locally refining and de-refining the mesh to capture flow-field phenomena of interest, adaptive methods make standard computational methods more cost effective. Similar issues also exist for problems in which the amount of computation associated with each mesh element changes over time. For example, in particles-in-cells methods that advect particles through a grid, large temporal and spatial variations in particle density can introduce substantial load imbalance. In both of these types of applications, it is necessary to dynamically load balance the computations as the simulation progresses. This dynamic load balancing can be achieved by using a graph partitioning algorithm. In the case of adaptive finite element methods, the graph corresponds to the mesh obtained after adaptation, whereas in the case of particles-in-cells, the graph corresponds to the

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original grid with properly adjusted vertex weights to reflect the particle density. We will refer to this problem as *adaptive graph partitioning* to differentiate it from the static graph partitioning problem that arises when the computations remain fixed. Adaptive graph partitioning shares most of the requirements and characteristics of static graph partitioning but also adds an additional objective. That is, the amount of data that needs to be redistributed among the processors in order to balance the computation should be minimized.

Traditional static and adaptive graph partitioning formulations are not general enough to ensure the efficient execution of multi-phase computations on high performance parallel computers. The reason is that these applications require the computation of partitionings that satisfy more than one balance constraint, while traditional graph partitioning techniques can balance a single constraint only. We refer to this as the *multi-constraint graph partitioning problem*. As in the case of a single balance constraint, multi-constraint partitioning problems can be either static or adaptive in nature. Recently, a number of multi-constraint graph partitioners have emerged that are able to compute partitionings that satisfy a number of balance constraints while also minimizing the edge-cuts. These have seen a great deal of success in meeting the partitioning requirements for a wide range of multi-phase and multi-physics simulations. However, many of these schemes are limited for two reasons. (i) They are serial in nature. (ii) They address only the static multi-constraint partitioning problem (and not the adaptive multi-constraint partitioning problem).

In this talk, we describe formulations of the multi-constraint graph partitioning problem that can be used to model the partitioning requirements of a wide range of scientific simulations. We also present the parallel formulations of static and adaptive multi-constraint graph partitioning algorithms and give experimental results from 128 processors of a Cray T3E. We show that our parallel algorithms are able to efficiently compute partitionings of similar quality to serial multi-constraint algorithms, can scale to large graphs, and are very fast. For example, our static multi-constraint partitioner is able to compute a three-constraint 128-way partitioning of a 7.5 million node graph in about 7 seconds on 128 processors of a Cray T3E. This parallel partitioner is used as the key component of an adaptive multi-constraint algorithm. We show that the adaptive scheme is able to quickly and robustly balance multi-constraint partitionings, while also minimizing the edge-cut. Furthermore, this scheme results in significantly lower data redistribution costs than the static algorithm.

THE *hp*-VERSION OF THE DISCONTINUOUS GALERKIN TIME-STEPPING METHOD

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We introduce and analyze the *hp*-version of the Discontinuous Galerkin (DG) time-stepping method for non-linear initial value ODEs and for linear parabolic PDEs. New a-priori error estimates are presented which are completely explicit in the time steps, in the approximation orders, and in the regularity of the exact solution. While these estimates allow us to recover the optimal convergence rates in the time step Δt , they also show that the DG method converges if the order $r \rightarrow \infty$ and the time step is kept fixed. We are able to prove that this *p*-version DG approach gives spectral accuracy for solutions with smooth time dependence, i.e., the convergence rates are of arbitrarily high algebraic order. Moreover, for entirely analytic solutions, this *p*-version convergence is of exponential order.

Typically, solutions of initial value problems become very smooth in time (in many cases even analytic) after a possibly non-smooth initial phase induced, e.g., by incompatible initial data or by piecewise analytic forcing terms. The resolution of such solution behavior in time requires appropriate *h*- and *p*-refinement towards the singularity. In conjunction with geometric time steps and linearly increasing approximation orders, we show the *hp*-version of the DG method can approximate temporal singularities at exponential rates of convergence.

A complete *hp* discretization in time and space is discussed for linear parabolic problems. At each time step a system of possibly singularly perturbed reaction-diffusion equations is solved. If the *hp*-Finite Element Methods for these spatial problems take into account certain mesh design principles using anisotropic and geometric mesh refinement techniques, exponential rates of convergence in time and space can be achieved.

All theoretical results are confirmed and illustrated in a series of numerical examples.

ON A HIGH ORDER BOUNDARY ELEMENT METHOD

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The aim of our contribution is an efficient method which is also parallelizable providing the following features:

1. The computation of boundary displacements and boundary tractions with high accuracy either on the whole boundary curve ($n = 2$) or boundary surface ($n = 3$) or on an a-priori chosen subregion of the boundary and
2. the computation of displacements, strains and stresses near and up to the boundary with high accuracy.

Our method is based on the decomposition idea applied to rather different parts of the boundary integral equation method:

- a) the decomposition of the boundary curve or boundary surface into an overlapping geometric partition,
- b) the decomposition of the trial space for the desired quantities into a regular coarse grid space and local spaces on fine grids; and
- c) the additive decomposition of the boundary integral operators involved into a standard principal part and a smoothing remainder.

As an example of our approach we consider the Lamé equations of linearized isotropic homogeneous elasticity,

$$\mu \Delta \vec{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \vec{u} = \vec{0} \quad \text{in } \Omega,$$

where the Cauchy data on the boundary $\Gamma = \partial\Omega$ are the boundary displacements and the boundary tractions, respectively, i.e.

$$\vec{u}|_{\Gamma} = \vec{\varphi} \quad \text{and} \quad \lambda(\operatorname{div} \vec{u})\vec{n} + \mu \left\{ \frac{\partial \vec{u}}{\partial n} + \sum_{\ell} n_{\ell} \operatorname{grad} u_{\ell} \right\}|_{\Gamma} = \vec{\psi}.$$

Here the complete stress σ on the boundary is for two-dimensional problems given by

$$\sigma \vec{n}|_{\Gamma} = \vec{\psi} \quad \text{and} \quad \sigma \vec{t}|_{\Gamma} = (\vec{\psi} \cdot \vec{t})\vec{n} + \frac{1}{(\lambda + 2\mu)} \left\{ 4\mu(\lambda + \mu) \left(\frac{d\vec{\varphi}}{ds} \cdot \vec{t} \right) + \lambda(\vec{\psi} \cdot \vec{n}) \right\} \vec{t}$$

where near to the boundary we use Hadamard's natural local coordinates in normal and tangential directions of the boundary, respectively; in 2-d with arclength parametrization s of Γ and \vec{n} , the exterior unit normal, \vec{t} the unit tangent vector. For 3-d problems, on the boundary surface Γ , the approach is principally the same.

To exemplify our method, let us consider, for simplicity, the Dirichlet problem in plane elasticity where we have to solve a boundary integral equation for the Cauchy datum $\vec{\psi}$, e.g. the integral equation of the first kind,

$$(5) \quad V\vec{\psi} = \vec{F} := (\frac{1}{2}I + K)\vec{\varphi} \quad \text{on } \Gamma.$$

In a first step, this equation is solved on the whole boundary where we use coarse grid trial functions on a regular grid associated with the mesh width H . In order to obtain high order convergence, we use the method proposed in [1], [2], where we approximate simultaneously tangential derivatives of $\vec{\psi}$ on the same grid which are used for higher order recovery. Since the coarse grid involves only a relatively small number of degrees of freedom, the coarse grid solution $\vec{\psi}_H$ approximating low frequencies, can be obtained with relatively small cost. Next we use the geometric decomposition together with an associated partition of the unity in order to localize the boundary integral equation. Now the fine grid solution on a window, i.e., on one of the subdomains $\Gamma_0 \subset \Gamma$, can be solved by using a local Galerkin method. In addition, we split the operator into a simplified principal part V_0 and a remainder that has a kernel which is less singular. Then we solve the local boundary integral equation for $\tilde{u} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_0)$ on the fine grid where the simplified operator, i.e.

$$(6) \quad \langle V_0 \tilde{u}, \tilde{v} \rangle = \langle V_0 \omega u_H, \tilde{v} \rangle - \langle \omega(Vu_H - F), \tilde{v} \rangle \quad \text{for all } \tilde{v} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_0)$$

defines the corresponding local influence matrix whereas the modified right-hand side contains all the global information in terms of the coarse grid solution, the corresponding residual and the remainder operator. Again we use simultaneous approximation of tangential derivatives and recovery for improving the efficiency of the local method. For smooth boundaries we present corresponding asymptotic error estimates which are related to the two-grid finite element approximations in [4]. For nonsmooth domains, the method can be modified appropriately and also performs astonishingly efficient [3].

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NON-CONFORMING hp MORTAR FINITE ELEMENT METHODS

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Non-conformity in the hp version can involve incompatibility in both the degrees and the meshes between adjoining subdomains. In this regard, the mortar finite element techniques have become very popular in joining together such incompatible hp discretizations. In this talk, we introduce some variants of the traditional mortar finite element and discuss the stability of these methods by introducing a new measure, that takes the place of the usual *inf-sup* stability constant. Our numerical results show optimality of the resulting non-conforming method for various h, p and hp discretizations, including the case of exponential hp convergence over geometric meshes. We also present results for the performance of these methods when the *stiffness* of the differential operator varies in different subdomains. Three-dimensional considerations suggest that the variants introduced will be much easier to generalize to arbitrary meshes than the traditional mortar finite element method. Some computational results for the Stokes problem and the mixed elasticity problem discretized by the hp mortar finite element method will also be presented.

STABLE AND EFFICIENT SPECTRAL METHODS IN
UNBOUNDED DOMAINS USING LAGUERRE
FUNCTIONS

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While the Legendre- or Chebyshev-spectral approximations for PDEs in bounded domains have achieved great success and popularity in recent years (see e.g. [3, 2]), spectral approximations for PDEs in unbounded domains have only received limited attention. Pioneer work on Laguerre approximation was developed in Gottlieb and Orszag [3] and Maday, Pernaud-Thomas and Vandeven [5]. Most of the previous numerical results used Laguerre polynomials which, from a theoretical point of view only provide meaningful results inside small intervals since the error estimate is obtained in weighted Sobolev spaces with an exponentially decaying weight, and from a practical point of view are not suitable for practical computations due to the extremely ill-conditioned behaviors of the Laguerre polynomials and of the Laguerre-Gauss-Radau quadrature formula. Furthermore, it was pointed out by Gottlieb and Orszag [3] that Laguerre polynomials/functions have very poor resolution properties when compared with other type of orthogonal polynomials. These are the three main factors which have limited the interests and developments on using Laguerre polynomials for problems in unbounded domains.

The aim of this work is to develop effective remedies for the aforementioned difficulties associated with Laguerre approximations. More precisely, we will propose uniformly convergent Galerkin approximations using Laguerre functions for elliptic equations in regular unbounded domains, and construct stable and efficient numerical algorithms for their practical implementations. Our theoretical error estimates and numerical experiments indicate that

- from both the theoretical and practical points of view, the Laguerre polynomials are generally not suitable for practical implementations;
- with a proper scaling, the Galerkin method using Laguerre functions provides a very efficient yet very simple way for solving problems in unbounded domains;
- the Laguerre-Galerkin approximation may converge exponentially for solutions which decay algebraically and monotonically at infinity. However, only algebraic convergence is possible if the solution *decays algebraically and oscillates* at infinity.

In summary, as demonstrated in this paper and in [4] where a similar Laguerre-Galerkin method is successfully used for the approximation of some nonlinear partial differential equations in semi-infinite intervals, we believe that properly formulated spectral methods using Laguerre functions are very valuable tools and viable alternatives to rational polynomials for problems in unbounded domains.

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ALGORITHM TO GENERATE P-VERSION MESHES FOR CURVED DOMAINS

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High-order (p -version) are capable of exponential rates of convergence [1]. This increased approximation power allows the use of coarse meshes in which relatively few elements cover large portions of the computational domain. Preserving the convergence rate for problems over curved domains requires that a curvilinear mesh geometry representation be used. This paper is concerned with the task of generating valid curvilinear meshes for general non-manifold solid models as defined in commercial geometric modeling systems.

CAD systems are becoming the mainstay of engineering design environment. The effective application of engineering analyses requires the ability to directly interact with these models. Over the past several years the technologies have been developed to support the automation of low-order finite element methods for objects defined in a solid modeling system. The most powerful of these capabilities directly access the boundary representation of the object to be analyzed and employs the functionality of the solid modeler to support the geometric operations needed by the mesh generator. For example the MEGA system [2] meshes general non-manifold geometric domains through a set of operators that have been directly linked with the geometry engines of the ACIS, Parasolid and Shapes modeling kernels used by many of the commercial CAD systems. From the standpoint of supporting engineering analysis, the importance of non-manifold solid models is that they can represent objects that are combinations of solids, surfaces and curves.

This paper addresses the additional capabilities needed to generate coarse curved meshes appropriate for p -version finite element methods. The methods to be presented will rely on a set of mesh modification operations to curve those mesh entities on the boundary of the model [3]. As with any operations that modify the geometry of the mesh, the process of curving mesh entities can convert valid straight-sided elements to invalid curved elements. Since the desired meshes can be coarse with respect to the local model curvature, the geometric changes associated with curving the boundary entities can be large. Therefore, the algorithm required to successfully create the curved meshes must include a complete set of mesh modification operators that must be applied in a carefully controlled manner. The overall algorithm to perform these modifications is:

```

Preprocess list of entities to be modified and determine target geometry
    while list of mesh entities not empty, loop over the mesh entities in the list
        determine allowed move that maintains a limit on mesh deformation in a
        single step
            if the current entity can accept the allowed move
                move to that location
                if the move completes the move to the target geometry remove entity from
                list
            else
                determine the first mesh entity preventing motion
                "analyze" current situation to determine "best" local modification
                perform selected local modification
                determine safe move distance and move
                determine new for the next step
            end if
        end while

```

The operators used in this process are local mesh modifications that focus on altering the mesh entities that constrain the desired mesh motion from being performed. The basic single step mesh modifications include mesh entity swap, collapse, split and geometry change operations. Often, it is not possible to attain the desired result without chaining a small number of these operators into compound operations.

The paper will present the technical details of a geometry-based mesh modification procedure for the generation of p-version meshes. Areas to be addressed include the geometric representation of the mesh entities, the mesh modification operators for curved meshes, procedures for evaluating the geometric properties of mesh entities, and the details of the mesh curving algorithm.

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**TRELLIS: A GEOMETRY-BASED ADAPTIVE ANALYSIS
FRAMEWORK FOR A MULTIPROCESSOR
ENVIRONMENT**

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Trellis is a geometry-based analysis framework for finite element and other types of numerical analysis, which is based on

- A set of geometry-based structures which can support direct linkage with company CAD information, and adaptivity without introducing geometric approximation errors.
- A careful decomposition of the geometry, physics, mathematical model, discretization and numerical methods into interacting classes that support multiphysics coupling.
- Adaptive control of the numerical solution process.
- Simulation automation supported by the MEGA geometry-based mesh generation procedures.

As the simulations become more detailed, the computational requirements can grow too large for effective solution in serial. In this case it becomes desirable to employ parallel computing techniques. Trellis makes use of the Rensselaer Partition Model (RPM) as the primary tool to deal with the parallel data management issues arising from the evolving nature of adaptive discretizations. The solution of the linear system of equations makes use of generic programming techniques allowing to template the parallel solution procedure over the serial implementation.

This paper will present progress to date on the development of the geometry-based parallel analysis framework Trellis. Conceptually Trellis is built on the view of an analysis as a transformation between three levels of description. The highest level description is that of the physical problem which is posed in terms of physical objects interacting with their environment. Since the goal of the analysis is to obtain reliable estimates of the response of the system the second level is a mathematical problem description that introduces some level of idealization, which also needs to be controlled to yield the desired accuracy. The third level is the numerical discretization constructed

from a mathematical problem that involves another set of idealizations that also need to be controlled.

Trellis represents a careful implementation of the overall structure of a numerical analysis process into a framework. It starts at the mathematical problem description providing classes that predefine the design so that the application programmer can concentrate on the specifics of the application. Key abstractions in Trellis are the analysis class that is being used to transform a mathematical problem into a discrete form. The discrete form is represented by the discrete system class that contains the overall system represented as contributions from system contributors (stiffness contributors, force contributors, and constraints). At this point the problem is described in an analysis-independent manner. Constructing the discrete system in parallel is based on utilizing RPM, which provides a notion of partitions, and partition boundaries describing the distributed entities of the discretization.

The solution of the linear systems of equations in parallel is based on generic programming techniques. The solver components are written to conform to specified requirements so that the actual parallel solver can be templated over the serial implementation of matrices, vectors, and operations on them, as well as a connectivity class describing the representation of the parallel distribution. This allows the extension of the solver part with new matrix and/or vector storage formats, without the need to specify parallel implementations for the algorithms.

MESH ADAPTION FOR IMPROVED GEOMETRY REPRESENTATION WITH HIGH ORDER ELEMENTS

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The development of unstructured spectral/ hp elements using elemental domains such as tetrahedrons and triangles has encouraged the use of automated mesh generation. However most mesh generators have been developed for use with linear elements.

Although all the problems associated with mesh generation for linear elements are inherent in mesh generation for high order elements the converse is not true. For example, the introduction of surface curvature in the elemental representation can lead to self intersecting elements which will have a vanishing or negative Jacobian thus a singular isoparametric mapping. This problem is also exaggerated by the fact that when using high order elements we typically want a much coarser discretisation.

Even when the surface is not curved the mapping from the parametric surface representation to the physical coordinates can generate problems as illustrated in figure 2. In this example we have represented the planar surfaces of a cube by an equispaced mesh of discrete points and a quadratic distribution of discrete points. In the case of the equispaced definition the mapping to the parametric space of the representation of the surface is isometric and we obtain the surface distribution of elemental points shown on the left. However in the case where we use a surface mapping with quadratic distribution of points we obtain the distorted surface reconstruction of the cube boundary shown on the right.

These problems have been highlighted from investigations in the field of computational haemodynamics which involve the generation of computational meshes of arterial bypass grafts. The paper will address the problems associated with surface modelling of highly complex geometries and discuss strategies to make the high order mesh generation and adaption more accurate and robust.

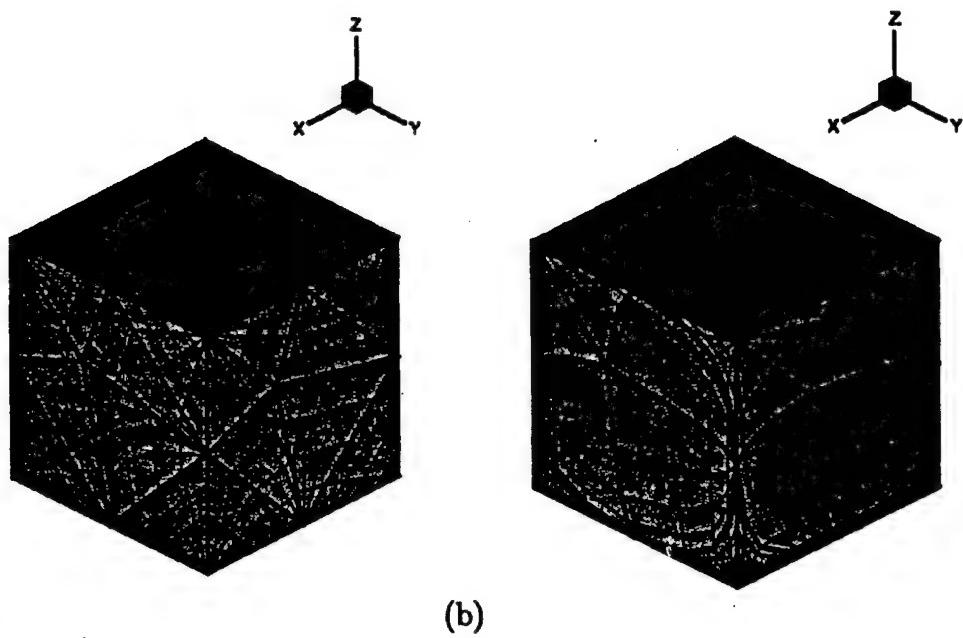


Figure 2: Deformation of surface element due to the mapping from the parametric space of the surface representation. The mapping in (a) is isometric and the mapping in (b) is not.

COMPUTATIONAL HEEMODYNAMICS IN ARTERIAL BYPASS GRAFTS

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The application of three-dimensional computational modelling to study the human



Figure 3: Wall shear stress distribution over a reconstruction of a coronary bypass

arterial system has become widespread over the last decade. This area presents a series of challenges in flow modelling due to pulsatile nature of the flow which is typically at a moderate Reynolds number regime where both viscous and inertial effects are important. Further we recognise that the flow geometries are very complex, the artery walls are distensible and the blood behaviour is non-Newtonian.

The good accuracy and phase properties of high order algorithms has made them particularly suitable for scientific studies of flow physics and therefore for understanding haemodynamics. Furthermore there is a strong body of evidence to show that disease processes such as myointimal hyperplasia and arteriosclerosis are related to wall shear stress. Therefore there is a need to accurately model both the primitive variables, v & p , and the derivative quantities associated with wall shear stress.

Figure 3 shows a computational reconstruction from an MRI image of an coronary bypass graft and the computed wall shear stress distribution. The paper will discuss how high order unstructured spectral/*hp* element methods are being used to model flows in arterial bypass grafts and the possible implication for interventional surgery.

**ADDITIVE SCHWARZ METHODS FOR THE
hp-VERSION OF THE BOUNDARY ELEMENT
METHOD FOR FIRST KIND INTEGRAL EQUATIONS
IN R^3**

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We consider screen problems (with a hypersingular or weakly singular integral equation on an open surface Γ) as model problems. For the hypersingular equation the preconditioner is based on a three-level decomposition of the underlying ansatz space, the levels being piecewise bilinear functions on a coarse grid, piecewise bilinear functions on a fine grid, and piecewise polynomials of high degree on the fine grid. We prove that the condition number of the preconditioned linear system is bounded by $\max_j (1 + \log \frac{H_j p_j}{h_j})^2$ where H_j is the diameter of an element Γ_j of the coarse grid, h_j is the size of the elements of the fine grid on Γ_j , and p_j is the maximum of the polynomial degrees used in Γ_j . For the weakly singular integral equation, where no continuity of test and trial functions across the element boundaries has to be enforced, the method works for non-uniform degree distributions as well. We comment also on various adaptive algorithms. Numerical results supporting our theory are reported.

RELIABILITY OF AN hp ALGORITHM FOR BUCKLING ANALYSIS

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Experimental determination of loads that can cause buckling and other failures is both expensive and unreliable. Mathematical models used in current engineering practice are, on the other hand, based on dimensionally reduced descriptions of an elastic body. The assumptions necessary for such dimensional reduction to hold can often result in a large, unknown modeling error when non-zero initial stress states are present, leading to inaccurate failure predictions.

A linearized model for buckling and stress-stiffening by Szabo and Kiralyfalvi [1] has been recently implemented in the hp code STRESSCHECK. This model does buckling analysis for the fully three-dimensional problem at hand, rather than some asymptotic (dimensionally reduced) limit. It finds the smallest positive multiple λ of an existing (pre-buckling) stress state σ_0 that will result in buckling. The use of the hp method enables solutions over singular domains to be well approximated, and ensures that no locking takes place even when the domain is very thin.

However, a potentially serious danger of the method is that it characterizes λ as the lowest positive spectral value of a problem of the form

$$A - \lambda B = 0$$

where A and B are *both* second-order differential operators. The finite element approximation of such (non-compact) problems could (at least theoretically) be very ill-behaved, due to the presence of spurious approximate eigenvalues, which can completely pollute the results.

We show that (1) spurious eigenvalues are *absent* for problems of engineering interest for which the pre-buckling stress σ_0 is bounded (2) spurious eigenvalues are *always present* for problems where σ_0 is unbounded. For the latter case, we demonstrate how the reliability of the computations can still be assessed, using the eigenvectors (the buckling shapes). We describe our results below.

1. Absence of spurious eigenvalues for bounded σ_0

By considering the asymptotics of a plate of thickness d , we show that no polluting eigenvalues will be observed in an interval $[C_1 d^{-2}, C_2 d^{-2}]$ about the origin. Hence,

spectral pollution will not occur for the eigenvalues of interest, provided the domain is thin. (The thin case is the case of engineering interest.)

Again considering the asymptotics of a plate, we investigate the effect of various initial stress states. We show that the only case where spurious eigenvalues could occur is the one where the initial stress is of pure bending type. This case is, however, uninteresting from the point of view of applications. For all initial stresses that occur in applications (these can be decomposed into a bending and a non-zero membrane stress), spectral pollution does not occur in the region of interest.

Our asymptotic analysis has also established that boundary layers occurring in the solution do not lead to adverse affects in the approximation of the desired eigenvalues.

Finally, we characterize the initial stresses that lead to pollution-free eigenvalue approximations when parts of the domain are not thin.

2. Reliability of computations for unbounded σ_0

For domains with corners, or domains with a change in the type of boundary conditions, it is well-known that the stresses can have $r^{-\alpha}$ singularities. We show that in this case, the value 0 always lies in the continuous spectrum of the problem. This causes the numerical approximations (which appear as a cluster of spurious eigenvalues) to converge to 0. In applications, however, one is interested in the lowest eigenvalue of a family related to the spectrum of a limiting problem (the $d = 0$ case). We characterize the values of α for which this limiting spectrum is still well-defined, and well-behaved.

We show computationally determined buckling shapes associated with spurious and non-spurious modes. It is seen that for spurious modes, the eigenfunction is almost constant over the whole domain, except for the region surrounding the point of singularity. These can therefore be easily distinguished from non-spurious modes, where the eigenfunction is 'global' as opposed to 'local.' Moreover, although the spurious eigenvalues converge exponentially to 0, the limiting eigenvalues of interest converge even faster, and hence can be recovered, as long as they are smaller than the spurious eigenvalues.

Two important facts that emerge from our investigation are: (1) The degree of localness of the eigenfunctions can be an effective tool in assessing the reliability of the computations (2) There is often an optimal level of layer refinement, over which spurious eigenvalues can arise. Our results therefore indicate that subject to the above limitations, the hp method in [1] is robust for engineering applications.

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THE CHALLENGE OF OBTAINING RELIABLE RESULTS IN CONTACT PROBLEMS

Yehuda Volpert

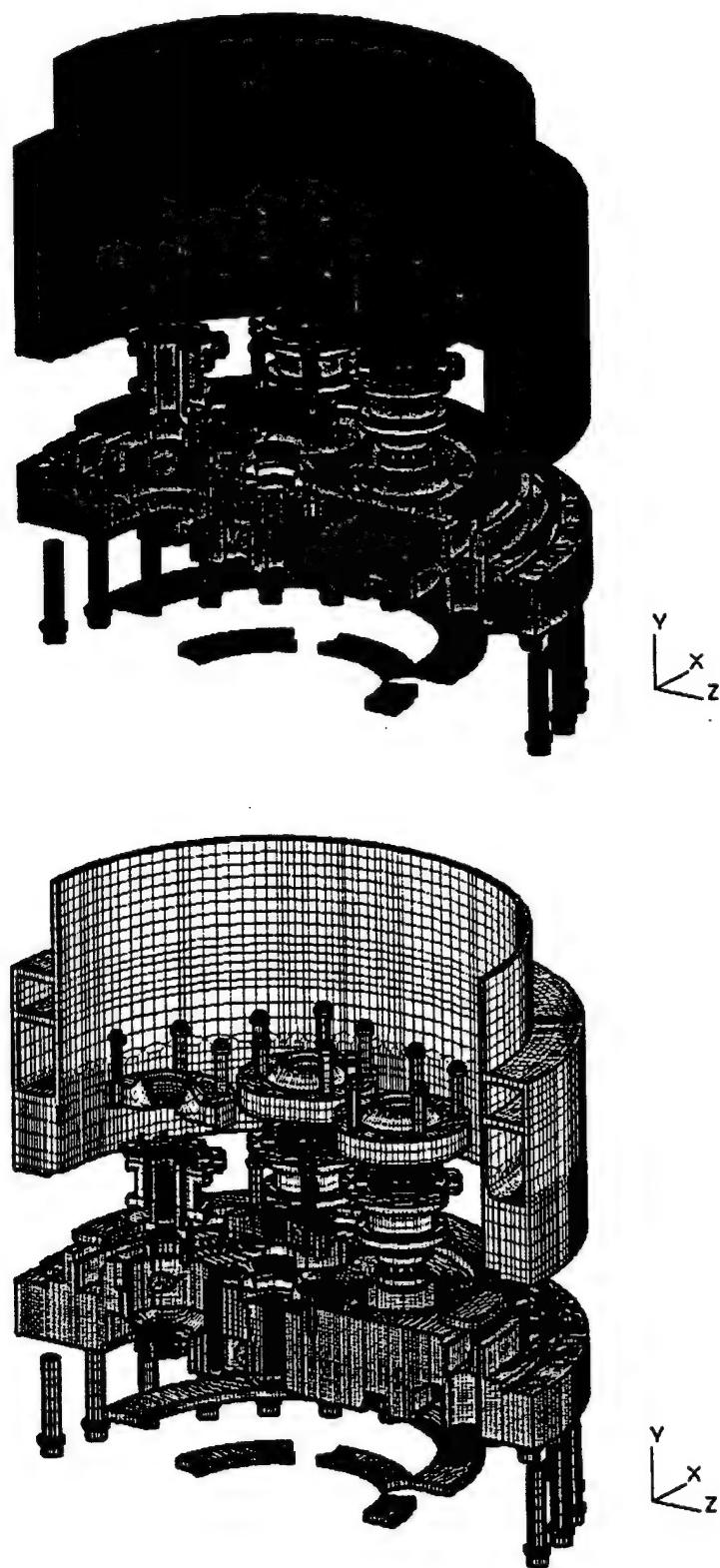
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For the past few decades more and more industries are using computer simulations to establish the response of the structure to loads associated with operational conditions. The advantage of the computer simulation over a more traditional approach (make and break) is obvious: companies save millions of dollars when costly experiments are replaced by analyses. However, computer simulation of the behavior of a real structure is not a trivial task. The finite element method, provided in a number of commercial codes, is a commonly used numerical simulation tool. Although the finite element tool is the most popular simulation tool used in industry, the capabilities and limitations of the tool are seldom well understood by the tools' users. It is "NEWS" to many users that the finite element method is just a numerical approximation to the solution to a particular set of partial differential equations with prescribed boundary conditions—in structural analysis, of course these are the equations of equilibrium (motion) for a single body. Real world structures, however, consist of many parts, and the task is to assess the behavior of the entire assembly. Mathematical modeling of the connection between different parts becomes an essential part of many simulations. A variety of contact algorithms are implemented in the commercial codes that perform this task. It is however questionable whether the available tools provide reliable results which can be used to closely approximate the response of the structure. In this paper finite element models of a structural assembly that is a part of a combustor area in an industrial turbogenerator are analyzed, Figure 1. The connectivity between different parts in this model is often simulated with contact or gap elements. In this paper, a small subset of this larger more complicated problem will be analyzed. Results obtained using *h*-version commercial code ANSYS are compared with results using the *p*-version code StressCheck. The goals of computation and the quality of the results are discussed.

Figure 1 Combustor. Top-Solid Model. Bottom-Mesh



APPLICATION OF NODAL UNSTRUCTURED SPECTRAL ELEMENTS TO MAXWELL'S EQUATIONS

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Introduction

We propose to demonstrate the application of the discontinuous Galerkin method (DGM) to Maxwell's equations using new developments for nodal unstructured spectral elements in two and three dimensions. Specifically we will discretize the evolution equations for the magnetic field and electric field vectors in a medium of, possibly sharply, variable permeability and permittivity. This is the typical physical description for the electromagnetic fields emitted by a mobile phone and their transmission in the air and user's skull cavity. There are currently significant debates about the possible harm caused by prolonged use of mobile phones. We wish to demonstrate that using DGM on unstructured grids will be able to give accurate results for high frequency electromagnetic waves passing through such sharply heterogeneous materials in complex geometries.

Proposed Numerical Studies

The examples we will use are scattering by perfectly conducting bodies, scattering by dielectrics, and scattering by sharp points (i.e. a plane wave incident on a cone). These last two are most challenging for spectral methods, where the non-smoothness in the equations and boundary conditions can cause $O(1)$ errors. We will show that the DGM handles these cases efficiently and robustly. We will also show preliminary results for scattering from an F15 geometry.

Example: There is an exact Mie solution for scattering of incident plane waves by a perfectly conducting, infinitely long cylinder. The exact solution for the electric field component is:

$$E_z^{sc}(\rho, \phi) = -E_0 \sum_{n=-\infty}^{\infty} i^{-n} J_n(ka) \frac{H_n^{(2)}(k\rho)}{H_n^{(2)}(ka)} e^{in\phi}$$

in terms of (ρ, ϕ) polar coordinates.

We have already run this simple simulation in a domain of approx 16 diameters, with a radial ABC. We show in figure 4 the numerical solution at $t = 30$ using the DGM approach. This is visually indistinguishable from the exact solution. However

in the test of L_2 and L_∞ error (not including the ABC region) we see that the error saturates at a level of 10^{-4} . This saturation is due to reflections from the ABC and in the proposed paper we intend to investigate how to modify this and alternative ABCs to reduce back scattering from far-field boundaries. We will demonstrate alternative results using a PML (perfectly matched layer) formulation in order to improve these results.

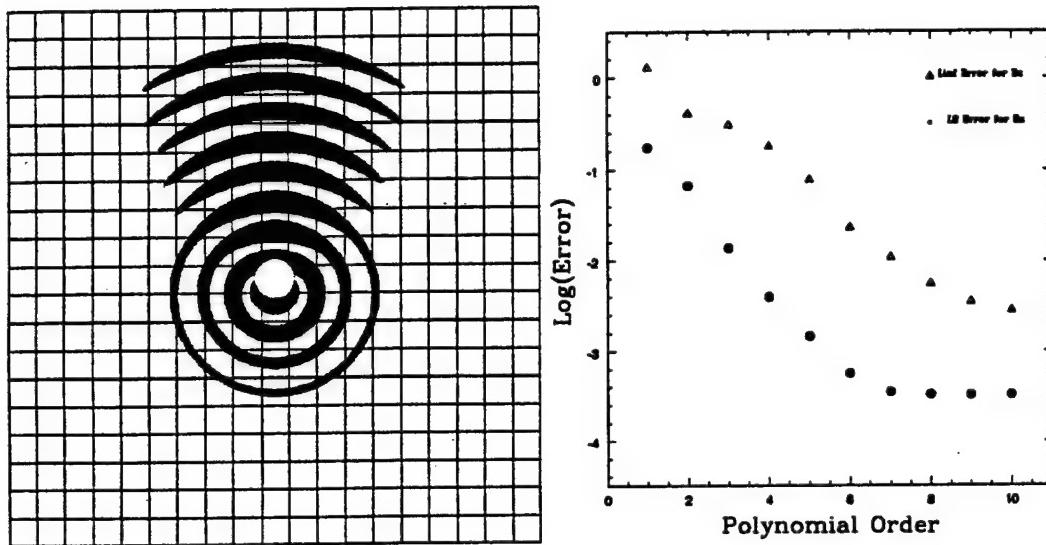


Figure 4: Left: DGM solution with ABC and $p=10$ at $t=30$ for scattering of a plane wave incident on a perfectly conducting cylinder. Right: Convergence plot for L_2 and L_∞ error for the DGM approximation compared with the Mie solution at $t = 30$ as a function of polynomial order.

DIAGONAL MASS MATRIX SPECTRAL METHODS FOR TRIANGLES AND SIMPLICES

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The classical formulation of the spectral element method based on the technique of Maday and Patera depends on the choice of functional space, grid points, and method of integration. On the quadrilateral, the preferred space is a tensor product basis of Legendre polynomials,

$$(7) \quad \mathcal{P}_{\square}^2 = \{\text{span } (x^n y^m) | 0 \leq m \leq N; 0 \leq n \leq N\}.$$

The grid points are chosen as the Gauss-Lobatto-Legendre points, which lead to an accurate, well-behaved interpolation, and subsequently well-behaved derivatives. The choice of points is also important for coupling elements together easily. By choosing a quadrature slightly less accurate than the optimal Gaussian, one achieves a computationally efficient diagonal global mass matrix for explicit methods.

On the triangle the preferred space is,

$$(8) \quad \mathcal{P}_{\Delta}^2 = \{\text{span}(x^n y^m) | 0 \leq m, n \leq K; m + n \leq K\}.$$

The space \mathcal{P}_{Δ}^2 was also used by Dubiner who reintroduced an orthogonal basis on the triangle, which was first presented by Koornwinder. Dubiner also introduced a warped tensor product quadrature set for the triangle. However, this quadrature set is oversampled which leads to a dense global mass matrix. This is much more expensive than for the cleverly implemented explicit method on quadrilaterals.

Triangle Functional Space

On the quadrilateral the favorite basis functions are eigenfunctions of a singular Sturm-Liouville problem, therefore, by the standard integration by parts arguments, the convergence rate for the approximation of smooth functions depends on the regularity of the function being approximated and not on any special conditions at the element boundaries. It has been known for quite some time that similar problems exist for the triangle. We now know the following about simplices:

1. We now know the Koornwinder polynomials are the eigenfunctions of a singular Sturm-Liouville problem on the simplex. Therefore, the convergence rate for the approximation of smooth functions depends on the regularity of the function being approximated and not on special conditions at the element boundaries [4].
2. Unlike the case on the line, we now know the eigenvalues come in eigenspaces. The commonly used triangle truncation is a union of these eigenspaces for all eigenvalues less than a constant. This truncation is invariant under the natural symmetry group of the triangle [4].
3. We now know Jackson and Berstien type inequalities for these spaces [1].

Triangle Interpolation Points

There has been recent work on computing optimal interpolation points on the simplex. Each set of points has interesting properties, but we prefer Fekete points, as in [2], for the following reasons:

1. On the $[-1, 1]$ interval, Fekete points are the Gauss-Lobatto points.
2. On the square, Fekete points are the tensor product of Gauss-Lobatto points, Thus, the conventional spectral element method can also be considered to be a Fekete point method.
3. Under suitable assumptions we can show that the Fekete points along each edge of the triangle are Gauss-Lobatto points making the triangles and quadrilaterals naturally conform.

Method

We introduced the Fekete point spectral element method in [3]. There we showed spectral convergence for linear advection problems. In this work we answer some key remaining questions about the method:

1. Will the points integrate twice the space like Gauss-Lobatto points on the quadrilateral? Does the integration preserve integration by parts?
2. Is the method stable?
3. Is the method as efficient as that on the quadrilaterals?
4. Does the method perform as well for non-linear problems?

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EXTRACTING EDGE FLUX INTENSITY FUNCTIONS FOR THE LAPLACIAN USING p-FEM

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The solution to the Laplace operator in three-dimensional domains in the vicinity of straight edges is presented as an asymptotic expansion involving eigen-pairs with their coefficients called edge flux intensity functions (EFIFs). The eigen-pairs are identical to their two-dimensional counterparts over a plane perpendicular to the edge. The general form of the asymptotic expansion is:

$$\begin{aligned} u(r, \theta, x_3) &= \sum_{i=1}^{\infty} S_i(r, \theta, x_3) \\ &= \sum_{i=1}^{\infty} a_i(x_3) r^{\alpha_i} s_i(\theta) + c_{i1} \partial_3^2 a_i(x_3) r^{\alpha_i+2} s_i(\theta) + c_{i2} \partial_3^4 a_i(x_3) r^{\alpha_i+4} + \dots \end{aligned}$$

where x_3 is the coordinate along the edge, and r, θ are cylindrical coordinates in a plane perpendicular to the edge. $\alpha_{i+1} \geq \alpha_i$ are called edge eigen-values, $a_i(x_3)$ are analytic in x_3 and are denoted by edge flux intensity functions (EFIFs). $s_i(\theta)$ are analytic in θ , called edge eigen-function and c_{ij} are given constants.

Extraction of EFIFs, cannot be obtained in a straightforward manner over this two-dimensional plane. Consequently, a special method based on L^2 projection and Richardson extrapolation is presented for point-wise extraction of EFIFs from p-finite element solutions. The mathematical analysis is demonstrated by numerical experimentation. A similar but more efficient method, based on "energy projection", for extracting EFIFs is proposed. The details are provided in [1].

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THE SOLUTION OF VISCOELASTIC PROBLEMS BY THE MIXED hp-VERSION OF THE FINITE ELEMENT METHOD

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Viscoelastic behavior has been become of great importance with modern developments in technology, such as the demands on high temperature behavior of materials for gas turbines, jet engines and power plants, and also with extensive studying of biomaterials. However, the comprehensive understanding of the mechanics of viscoelastic problems is still hindered by a lack of reliable numerical simulation methods and experimental data for the characterization of material properties. So far, most of the procedures implemented in various finite element software are based on the traditional h-version of the finite element method. These procedures usually do not provide for control of the solution quality, making it difficult to estimate and control discretization errors. Proper control of discretization errors is crucial for the selection of the proper constitutive laws and the determination of the coefficients in the constitutive laws.

Our goal is to develop a stable and reliable algorithmic procedure for the numerical simulation of viscoelastic solids which allows estimation and control of the errors of discretization, therefore the material properties can be obtained efficiently using optimization algorithms that minimize the difference between experimental measurements and finite element predictions. Towards this end we have developed a new algorithmic procedure to solve viscoelastic problems using the mixed formulation. The control of spatial discretization errors is based on the hp-version of the finite element method in which the displacement and stress fields are approximated separately. The control of time integration errors is based on finite difference method. The algorithm has been implemented into a research code for planar and axisymmetric elements. Both linear and nonlinear constitutive laws are incorporated. For linear viscoelastic materials, the general differential constitutive equation is used. For nonlinear viscoelastic materials, several empirical creep laws are used. Small displacement and small strain conditions are assumed. Numerical results are presented for benchmark problems which show the robustness of the formulation.

Numerical testing has been performed with the objective to investigate the locking-free property of the mixed formulation for nearly incompressible materials. The results show that the method is free from locking and performs well for incompressible materials.

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Attendees

| Title | Last | first | Institution |
|-------------|---------------|-------------|--|
| 1 Dr. | Actis | Ricardo | esrd, inc. |
| 2 Dr. | Adjerid | Slimane | Virginia Tech. |
| 3 Prof. | Ainsworth | Mark | Strathclyde University |
| 4 Mr. | Amar | Gal | Ben-Gurion University of the Negev |
| 5 Dr. | Andersson | Borje | The Aeronautical Research Institute of Sweden |
| 6 Dr. | Ariosto | Jorge | Vanderbilt University |
| 7 Dr. | Auteri | Franco | Politechico di Milano |
| 8 Dr. | Azaiez | Mejdi | Universite Paul Sabatier De Toulouse |
| 9 Prof. | Babuska | Ivo | University of Texas, Austin |
| 10 Mr. | Basu | Prodyot | Vanderbilt University |
| 11 Prof. | Belytschko | Ted | Northwestern University |
| 12 Dr. | Ben Belgacem | Faker | Universite Paul Sabatier De Toulouse |
| 13 Prof. | Bertoti | Edgar | University of Miskolc |
| 14 Dr. | Bognar | Gabriella | University of Miskolc |
| 15 Mr. | Bowden | Dave | Boeing Company |
| 16 Prof. | Brenner | Susanne | University of South Carolina |
| 17 Prof. | Cai | Wei | University of North Carolina at Charlotte |
| 18 Lt. Col. | Canfield | Bob | AFOSR |
| 19 Dr. | Cao | Thang | The University of New South Wales |
| 20 Dr. | Carpenter | David | Bechtel Bettis, Inc. |
| 21 Dr. | Chan | Eduardo | PTC |
| 22 Dr. | Chilton | Lawrence | Air Force Institute of Technology |
| 23 Ms. | Chow | Sieng Khuan | Nanyang Technological Univ. |
| 24 Dr. | Chrisochoides | Nikos | University of Notre Dame |
| 25 Dr. | Cugnon | Frederic | University of Liège |
| 26 Mr. | Davis | Stephen | ARO |
| 27 | de Pennington | Alan | Leeds University |
| 28 Prof. | Demkowicz | Leszek | University of Texas, Austin |
| 29 | Dennison | Randy | esrd, inc. |
| 30 Dr. | Devine | Karen | Sandia National Laboratories |
| 31 Dr. | Dey | Saikat | SFA Inc. |
| 32 Dr. | Engelstad | Steve | Lockheed Martin Aeronautics Company - Marietta |
| 33 Prof. | Flaherty | Joseph | Rensselaer Polytechnic Institute |
| 34 Dr. | Gerdes | Klaus | Chalmers University of Technology |
| 35 Mr. | Gierer | Joe | Emerson Tool Company |
| 36 Dr. | Guo | Ben Yu | Shanghai Normal University |
| 37 Dr. | Guo | Benqi | University of Manitoba, Canada |
| 38 Dr. | Guo | Xian | Washington University |
| 39 Mr. | Halley | Jeremiah | Boeing Company |
| 40 Dr. | Halpern | Marc | D.H. Brown Associates |
| 41 Dr. | Heuer | Norbert | Bremen University, IWD |
| 42 Dr. | Holzer | Stefan | Universitaet Stuttgart |
| 43 Mr. | Honeycutt | Kyle | APES, Inc. |
| 44 Mr. | Iarve | Endel | University of Dayton |
| 45 Mr. | Jackson | Timothy | Advanced Enterprise Solutions |
| 46 Dr. | Karniadakis | George | Brown University |
| 47 Dr. | Katz | Norman | Washington University |
| 48 Dr. | Kopriva | David | The Florida State University |
| 49 | Krause | Roland | Washington University |
| 50 Mr. | Leist | Ric | Cessna Aircraft Company |
| 51 Dr. | Maischak | Matthias | University of Hannover |
| 52 Dr. | Maunder | Edward | University of Exeter |

| | | | | |
|----|-------|---------------|----------------|---|
| 53 | Ms. | Metzler | Sandra | Washington University |
| 54 | | Mitchell | John | |
| 55 | Mr. | Miyake | Masa | Applied Design Analysis Corporation (ADA) |
| 56 | Mr. | Myers | Kent | esrd, inc. |
| 57 | Mr. | Nagengast | Andreas | INA Motorenelemente |
| 58 | Prof. | Oden | Tinsley | University of Texas, Austin |
| 59 | Prof. | Oh | Hae-Soo | University of North Carolina at Charlotte |
| 60 | Dr. | Paczelt | Istvan | University of Miskolc |
| 61 | | Parashar | Manish | |
| 62 | Prof. | Patra | Abani | Univ. of Buffalo, SUNY |
| 63 | Dr. | Paydarfar | Saeed | AES (Advanced Enterprise Solutions) |
| 64 | | Peano | Alberto | ISMES |
| 65 | Dr. | Peters | David | Washington University |
| 66 | Dr. | Prost-Domasky | Scott | APES, Inc. |
| 67 | Dr. | Rank | Ernst | Tu Muenchen |
| 68 | Mr. | Reid | Paul | Boeing Company |
| 69 | Ms. | Riviere | Beatrice | University of Texas, Austin |
| 70 | Dr. | Rodin | Ervin | Washington University |
| 71 | Dr. | Saendig | Anna-Margarete | University Stuttgart |
| 72 | Dr. | Sahrmann | Glenn | UNIGRAPHICS |
| 73 | Dr. | Schiermeier | John | MSC Software Corp. |
| 74 | | Schloegel | Kirk | Univ. of Minnesota |
| 75 | Dr. | Schoetzau | Dominik | University of Minnesota |
| 76 | Dr. | Scholz | Dominik | Tu Muenchen |
| 77 | Prof. | Schwab | Christoph | ETH Zurich |
| 78 | Dr. | Seshaiyer | Padmanabhan | Texas A&M University |
| 79 | Dr. | Shen | Jie | Penn State University |
| 80 | Prof. | Shephard | Mark | Rensselaer Polytechnic Institute |
| 81 | | Sheppard | John | esrd, inc. |
| 82 | Dr. | Sherwin | Spencer | Imperial College |
| 83 | Dr. | Stein | Erwin | University of Hannover |
| 84 | Dr. | Stephan | Ernst | University of Hannover |
| 85 | Dr. | Suri | Manil | UMBC |
| 86 | Dr. | Szabo | Barna | Washington University |
| 87 | Dr. | Szabo | Tamas | Hungarian Academy of Sciences |
| 88 | Dr. | Toselli | Andrea | Courant Institute |
| 89 | Dr. | Volpert | Yehuda | ANALYSIS & DESIGN APPLICATION CO. |
| 90 | Dr. | Warburton | Tim | Brown University |
| 91 | Dr. | Wendland | Wolfgang | Universitaet Stuttgart |
| 92 | Mr. | Wheeler | Michael | TechNet International Corp. |
| 93 | Prof. | Widlund | Olof | Courant Institute |
| 94 | Dr. | Wingate | Beth | Los Alamos National Laboratory |
| 95 | Dr. | Yosibash | Zohar | Ben-Gurion University of the Negev |
| 96 | Mr. | Young | Keith | Boeing Company |
| 97 | Ms. | Zhang | Li | Washington University |
| 98 | | Zhang | Zhimin | Wayne State University |
| 99 | | | | |